

# Risky business

*Theoretical approaches to housing  
in the household portfolio*

Marius B. Østli



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# Preface

Late last year, when it was time for me to decide on a topic for my Master's thesis, I sat down at my usual seat in the university study hall and began pondering. I pondered on for quite a few weeks as I didn't feel obligated to go in any particular direction: I could pretty much write about anything. Soon enough though, I had filled a small notebook with dozens of ideas and possible project descriptions. Among the topics were motivational programmes, the economics of high-speed trains, a single world currency, the Oslo market for office space, European pension challenges, the possibility of perpetual economic growth, public management reform, the link between student effort and labour market performance, the value of a good reputation, and the economics of humane farming. I had certainly collected a "well-diversified" bundle of thoughts, but I wasn't too ready to write about any of them.

However, as I incessantly came up with new questions and ideas, I kept going back to the very first thought I wrote down, "the optimal timing of housing market entrance," which essentially formed the basis for this thesis. True, the final topic is the product of numerous alterations of the original idea, but the core question remains: how can I increase my lifetime wealth and utility by navigating optimally in the housing market?

Most of this paper is written at my desk in the research assistants' office at the Department of Economics, which I have been fortunate to have at my full disposal while working on my thesis. For this, and for being a great employer to a part-time working student, I am truly thankful to the department. I also thank ESOP – Centre for Equality, Social Organization, and Performance – for granting me a generous student stipend based on my original thesis proposal. Unforeseen, time-consuming technical challenges have unfortunately prevented me from writing as much about social inequality as I had intended, but the grant has nevertheless been a humbling and inspirational reminder throughout the process.

Although their web services otherwise are excellent, it wasn't always easy to find all the data I needed from Statistics Norway and from the Oslo Stock Exchange on my own. I therefore thank the good, helpful people at both institutions for providing me with useful time series and tables.

Lastly, the contributions of my supervisor, Øystein Børsum, have truly been indispensable and I am thankful for the helpful comments, the good advice, and the enthusiasm he has offered from beginning to end. I would also like to express my gratitude to Espen Henriksen for his invaluable help and support as I implemented my dynamic model, and to Diderik Lund for patiently taking the time to answer my many questions about the portfolio model.

Of course, all mistakes and shortcomings are mine and mine alone.

Marius Brekkeflat Østli  
Oslo, 2 May 2008

# Introduction

Buying a home and entering the housing market is typically the greatest investment a household will ever make. For most owners, the value of the home is many times that of all other asset holdings combined and therefore crucial in the household's accumulation of wealth over time. Since housing assets can serve as collateral, people are granted large mortgages, and thus even modest returns can yield great wealth boosts due to the sheer magnitudes of the investments. As such, no other form of investment can “pull people up” financially like housing can.

For example, 350 000 kroner invested in 1966 would on average had grown to more than 2.5 million 25 years later, in 1991. In real terms this amounts to a 49 % return on the housing investment. Further, by staying in the housing market, the 2.5 million 1991 dwelling would on average had increased in value to over 8.7 million in 2006.<sup>1</sup> The latter period was thus even more impressive return-wise than the former, producing an annual, real price growth rate of over 7%.

Of course, owning a home also produces housing benefits of great value to the owner-occupier. Everyone needs a place to live and owning a home secures a steady flow of dwelling services that would otherwise have had to be bought in the rental market. Housing is therefore often called a *dual asset* since its demand is driven both by the consumption- and the investment properties that come with it. As decisions regarding the housing asset profoundly influence the lifetime wealth and general utility of the individual household, a formal study of this topic should be most warranted. Drawing inspiration from the literature, the modest ambition of this thesis is therefore to probe into some aspects of households' resource allocation.

The presence of a rental market allows the divorce of housing consumption and housing investment. In theory we could have a household renting its desired quantity of dwelling services in the rental market and investing optimally in housing assets and other assets elsewhere. One could for example imagine “splitting” housing units into smaller pieces and trade these almost like stocks: rent would then yield dividend to the owners while the house price appreciation would stand for the capital gains (Caplin *et al.*, 1994). With such a market in place the household could fully separate consumption and investment with the

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<sup>1</sup>These historical returns are discussed in Chapters 2 and 1 respectively.

added benefit of diversification of the housing portfolio.<sup>2</sup> In practice however, this is difficult since such a market for “pieces of houses” does not exist. Neither does a real estate futures market (as proposed by Case *et al.* (1993)) nor more general housing price index funds. In light of this, it would be quite a coincidence if households’ holdings of housing are optimal from an investor point of view. In fact, Brueckner’s (1997) theoretical treatment shows that household portfolios, in which the amounts of housing are determined by consumption demand, are *inefficient* relative to those where investment and consumption are fully separated.

However, households seem to prefer owning to renting regardless of any possible discrepancy between consumption demand and investment demand. When asked in 1997, 95% of Norwegian households said they preferred owning to renting (Løwe, 2002a). Only after age 70 did this figure drop below 90% when 14% said they preferred some kind of renting arrangement. All other age groups went overwhelmingly for owning. Why this taste for homeownership? Said survey did not ask its respondents, but easy arguments might be the unwillingness to “throw money out the window” by being a renter, the desire to control the appearance and the fixtures of the dwelling, and perhaps a wish for the absence of a disagreeable landlord.

So what about actual behaviour? In Norway, around 50% of households owned the dwelling they lived in right after the First World War. The percentage increased steadily through the 20th century until stabilising just above 80% in the late 1990s, then a share among the highest in the world. Gulbrandsen (2004) comments that this widespread ownership is largely owed to active government policies and subsidies to self-owned units. Low-interest state mortgages were given to construction of new dwellings as long as they did not exceed certain limitations on size and building cost, and housing cooperatives were set up to provide members with cheap but high quality central dwellings. Further, the governments of the 1950s and 1960s emphasised low-cost self-owned housing for low-income earners rather than providing publicly owned rental units. This policy stood in sharp contrast to those of many European countries (Kurz and Blossfeld, 2004), and may have contributed substantially to the evident Norwegian taste for owner-occupancy.

The overall homeownership rate in Norway has been stably high for a while, but what if we consider this in relation to household characteristics? Not surprisingly, young households are less likely to be owners than older households, but even for the 25-29 cohort the ownership rate was 62% in 1997 (Gulbrandsen, 2004). In fact, roughly 70% of households are established as owner-occupiers before the age of 30 (Andersen, 2002). Further, wealthier households own considerably more housing than those of fewer means, and homeownership as such appears to be closely correlated with household income: in the bottom income quintile the homeownership rate is only 61% while it is 95% for the top quintile. In between, the rate rises consistently with income. Among older cohorts the

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<sup>2</sup>Say you reside in a dwelling you own a certain share of and, at the same time, own smaller pieces of other dwellings elsewhere.



ownership rate quickly reaches 90% before dropping slightly for the oldest.

It thus seem like a priority for people to get into the housing market as early as possible, which is an especially good idea if one believes real house prices are steadily increasing over time. We will note in the first chapter however, that this is definitely not always the case: true, real housing price has shown an upward trend in both the short- and the long-run, but there have been several periods of time where prices have fallen too over the years.

Moreover, the owning preference is rationalised financially as well: one unit of rental housing is more expensive than an equivalent unit of self-owned housing (Brueckner, 1997). That is, in principle, a dwelling's monthly market rent is higher than the same dwelling's monthly cost (mortgage interest payments, maintenance, etc.) had it instead been an owner-occupier residence. Why? Firstly, a landlord's rental income is taxed just as any other income, while the housing services produced by a self-owned dwelling are not taxed at all. Naturally, the landlord will roll these taxes over on the tenant's rent. Secondly, assuming renters will be less careful and vigilant about the dwelling's condition than homeowners, rental units will depreciate faster and require more maintenance, thus forcing landlords to add a premium to rents.

From any household's point of view, consumption and investment in the housing market is somewhat risky. While the renter faces the risk of increased rents, the owner is exposed to interest rate risk. If the home purchase is financed by a floating rate mortgage (which the vast majority of Norwegian purchases are), the monthly payments will depend to a considerable extent on fairly volatile interest rates. The latter effect has of course been especially visible in recent months when many low-income owners have had trouble keeping up with mortgage payments.

The general uncertainty of housing prices and the possibility of large fluctuations present two opposing "risks" for the renter and the owner. Whether the renter plans to become an owner in the future or not, a relative price increase on housing today means immediate redistribution of wealth from non-owners to owners. The renter thus becomes relatively poorer and it will be more difficult to enter the housing market later. If, on the other hand, there is a drop in real prices, the owner is worse off while the renter may enter the housing market at a relatively lower cost than those who entered earlier. The fact that housing investment usually entails purchasing a single, immobile unit amplifies the risk to the homeowner considerably. Even if the overall market is performing well, individual properties may very well suffer negative idiosyncratic shocks to their value. Further, a crucial characteristic of a housing investment is that it is illiquid, meaning that buying and selling is costly in terms of time and money.

With respect to the latter and to the size of unit investments, equity differs greatly from the housing asset. It is also what perhaps comes first to mind when we talk about "investment" as such. Although investments in the stock market are risky (potentially a lot riskier than in the housing market), the nature of this market allows the spreading of one's total investment over many sectors and companies so that the overall risk may be lowered substantially. Such diversification is impossible in the housing market with just one asset unit, thus

the inclusion of stocks in the household portfolio might mitigate one's exposure to risk. Further, with reference to the example above, stock market returns are usually no short of those in the housing market. For instance, say 100 000 kroner were invested in equity in 1966. By 1991 this would have amounted to, on average, a whopping 1.9 million. In real terms the 25-year return was nearly 430%. Almost shockingly, this pales in comparison to the nominal value in 2006 had the investment remained in the market: 24.4 million kroner. Clearly, equity holdings are something any optimising household should seriously consider.

In the following we will consider the household both as an investor and as a consumer, but abstract from the rent-own decision for simplicity. In Chapter 1 the Norwegian household portfolio will be put under scrutiny together with the performance of corresponding assets. Properties of housing and stocks as assets will be discussed, naturally with weight on the former. The key question that emerges is that of how to optimally allocate one's resources and how to best compose a portfolio of assets when housing is explicitly included on the list of available investments (as opposed to taking the housing holdings as exogenously given). Given the history of actual asset returns in Norway, are the households' actual portfolio compositions close to being "optimal?" Are we perhaps able to explain the observed behaviour by a couple of simple models? How do optimal allocations evolve and change over time and over the households' age?

We will try to answer these questions in Chapters 2 and 3 where we employ a static model and a dynamic model respectively. The prescriptions of these will presumably contrast quite notably since their stylisations of the world differ in key areas. It will hopefully be an interesting exercise to investigate which one comes closest to our observations and how they each rationalise their advice on optimal allocations.

Chapter 4 briefly summarises the findings and approaches.

# Chapter 1

## Asset returns and portfolio observations

If our goal is to talk about how households *should* behave when faced with different financial- and real investment opportunities, we must first sort out the properties of these alternatives. A natural strategy will then be to look at how the various assets have performed in the past.

Further, advices on how people should behave are of little interest if we cannot relate these prescriptions to how people actually allocate their resources. This chapter seeks to address both these topics, starting with the latter.

### 1.1 Observed portfolios

To get an idea of households' actual portfolios we will follow Harding *et al.* (2004) and employ data from the *Income Distribution Survey 2002* from Statistics Norway.<sup>1</sup> This is the most recent survey of its kind and it is based on interviews with, and tax returns of, over 22 000 households in Norway. Table 1.1 reproduces the survey's findings of the average household property account.<sup>2</sup>

In the the models of the next chapters we will assume that the household may hold the following assets; housing, stocks, and positive or negative amounts of an interest-yielding risk-free asset. The data from Table 1.1 must therefore be rearranged into these categories for comparison.

Total real capital is in the table composed of real properties and other types of capital owned by the household. The latter is mostly cars, boats, machines, furniture, farm animals etc., and will be ignored as the observed portfolio is established. Real properties are primary dwellings, holiday cabins, farm houses, business properties etc., and will hereafter be labelled "housing."

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<sup>1</sup>Available online: [http://www.ssb.no/emner/05/01/nos\\_inntektformue/](http://www.ssb.no/emner/05/01/nos_inntektformue/)

<sup>2</sup>The components of gross financial capital are found in Statistics Norway's *Statistical yearbook 2004* (Table 215) as the survey report does not include these details.

	NOK
1. Total real capital	302 000
of which:	
1.1 Real properties	236 400
1.2 Other capital	65 600
2. Gross financial capital	363 700
of which:	
2.1 Bank deposits	205 800
2.2 Unit trust fund (stocks)	13 400
2.3 Bond and money market funds	9 100
2.4 Foreign bank deposits, bonds, stocks	4 300
2.5 Registered securities	21 600
2.6 Unregistered securities	65 500
2.7 Other claims	44 100
3. Debt	495 500
Net wealth	170 200

Table 1.1: Average household property account in Norway 2002. Source: Statistics Norway's Income Distribution Survey 2002.

Unfortunately, the figure reported in Table 1.1 is the taxable value of the housing asset, not the actual market value. Taxable values tend to be much lower than the market values since they are usually based on appraisals or on property owners' own assessments. Statistics Norway has estimated the ratio of taxable value to market value by comparing the sales price and the pre-sale taxable value of individual dwellings. The result was, in both the 1999 and 2005 study, a ratio of 0.2.<sup>3</sup>

Rather than assuming that this ratio also holds for 2002, we will go back to the *Income Distribution Survey* and collect the taxable value of housing in 1999. This value, 190 000, is divided on the relevant ratio and multiplied with the housing price index. The latter amounts to 1.27 from 1999 to 2002 so that<sup>4</sup>

$$\text{Market value (2002)} = \frac{190\,000 \cdot 1.27}{0.2} = 1\,207\,135. \quad (1.1)$$

Next, gross financial capital can be decomposed as stocks, bonds and bank deposits. We will merge the latter two into a "risk-free" asset. Bank deposits and bond and money market funds (2.1 and 2.3) belong here. Unit trust fund holdings (2.2) are assumed to be stocks although these funds often include significant holdings of bonds as well. Registered securities (2.5) are those securities registered with the Norwegian Central Securities Depository in 2002. According

<sup>3</sup><http://www.ssb.no/emner/05/03/sbolig/>

<sup>4</sup>With (year) 2000 = 100, Statistics Norway reports the 1999 to 2002 price level ratio to be 111.4/87.7 = 1.27.

	NOK	Share
Housing	1 207 135	1.175
Stocks	91 790	0.089
Risk-free	-271 890	-0.264
Net wealth	1 027 035	1.000

Table 1.2: Average household portfolio in Norway 2002. Source: Statistics Norway and own calculations.

to the depository's own statistics, stocks make up 90% of households' registered securities while 10% are bonds.<sup>5</sup> Assuming that this division also holds for unregistered securities (2.6), holdings of bonds and stocks can be identified as 8 710 and 78 390 respectively in posts 2.5 and 2.6 combined.

Foreign bank deposits, bearer bonds and stocks (2.4) and other claims (2.7) will be ignored in the following due to their somewhat "foggy" definitions.

### 1.1.1 An undiversified portfolio

Table 1.2 presents the actual mean household portfolio in both kroner and as shares of household net wealth. It is now clear that the average household has a highly undiversified portfolio; the housing asset holdings are nearly four times greater than the holdings of the other two assets combined. In addition, the housing holdings are typically in one or two individual properties, not in, say, housing market funds. As we will discuss below, the idiosyncratic risks of homeownership should be quite substantial so that a portfolio like that in Table 1.2 is very risky for the household. Further we see that the stock market participation is generally very low with a stocks-to-housing ratio of just 0.076. This is known as the *portfolio choice puzzle* in the literature (Hu, 2005). Cocco (2004) explains this by pointing to the housing risk: since the market- and idiosyncratic risks of housing are so high, households compensate by holding risk-free assets, i.e., reducing the mortgage rather than investing in the equity market. Stocks are thus crowded out by housing holdings due to households' aversion to risk. This argument also explains why the average debt is so low.

Harding *et al.* (2004) equivalently calculate the average portfolios for five consecutive years, 1998-2002. The holding shares seem fairly constant over this period except for that of stocks. The holdings of stocks were almost twice as high in the late 1990s as in the early 2000s (relative to average net wealth). This can perhaps be owed to the post-11 September 2001 downturn in the global stock markets: intuitively, poorer equity returns should have caused people to substitute away from stocks, at least temporarily. It may be a good idea to keep this in the back of our heads in the following: the share of stocks may be artificially low in the 2002 survey data.

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<sup>5</sup>Statistics available on the depository's website: <http://www.vps.no/>.

	-24	25-34	35-44	45-54	55-66	67-79	80-
Housing	2.848	2.607	1.492	1.132	0.959	0.801	0.736
Stocks	0.121	0.117	0.077	0.064	0.064	0.037	0.022
Risk-free	-1.969	-1.724	-0.568	-0.229	-0.023	0.162	0.243
Relative wealth	0.074	0.340	1.054	1.843	2.233	1.846	1.177

Table 1.3: Portfolio shares of age cohorts in Norway 2002. Source: Statistics Norway and own calculations.

### 1.1.2 Portfolios by age and wealth

The same procedure that led to Table 1.2 produces Table 1.3 when we consider property accounts by age cohorts.<sup>6</sup> The fourth row reports the ratio of cohort net wealth to population average net wealth. Unsurprisingly, young households start out with very little wealth, accumulate assets rapidly, and peak right before retirement. At all stages of life, housing accounts for a very large share of the average household's portfolio: from nearly three times net worth early in life to just under three quarters at the end. The average portfolio of Table 1.2 now looks almost cautious in comparison to the situation of the young households. This mirrors the holdings by age in Sweden in 1989 almost perfectly (Englund *et al.*, 2000), while American households exhibit even more extreme differences across cohorts; from a housing share of 3.5 times net wealth for the youngest, to just 0.65 for the oldest (Flavin and Yamashita, 2002).

Assuming most of the debt to be mortgage, the housing holdings are also heavily leveraged<sup>7</sup> early on at around 70% and 66% for the first and second cohorts.

The share of risky stock holdings is highest for the youngest group and declines consistently with age, in itself an indication of (relative) risk aversion increasing with age. On the other hand, one could perhaps make the argument that there has been increasing awareness, and marketing, of stock market investments over the years, making equity a more dominant asset in younger portfolios.

Table 1.4 reports portfolio shares by wealth quartile.<sup>8</sup> What is most striking

<sup>6</sup>The *Income Distribution Survey 2002* asked the respondent of the age of the "head" of the household (presumably this is the oldest member).

<sup>7</sup>By "leverage" we here refer to the magnitude of debt relative to housing holdings.

<sup>8</sup>Obtaining these portfolio shares was a bit more cumbersome than those in the other tables since Statistics Norway does not report full balance sheet details for the gross financial capital for wealth quartiles. Instead, everything but bank deposits is bagged in "other claims." Fortunately, financial capital details are available in a table reporting average balance sheets for persons (not households) by quartiles (the *Income Distribution Survey 2002* again). From these numbers we calculate the stocks-to-other claims ratio and bonds-to-other claims ratio for each quartile and assume that the per person data is representative enough for households. Multiplying the households' financial "other claims" with the relevant ratios yields the portfolio shares of Table 4.

Further, we assumed above (in the discussion of posts 2.5 and 2.6 in Table 1) that "securities" was 90% stocks and 10% bonds. This is true for the average investor, but probably not for, say, the average first or fourth quartile household. We will however let this inaccuracy

	1st	2nd	3rd	4th
Housing	-0.462	1.862	1.297	0.919
Stocks	-0.014	0.018	0.017	0.103
Risk-free	1.969	-1.477	-0.880	-0.022
Relative wealth	-0.119	0.484	1.340	3.431

Table 1.4: Portfolio shares by wealth quartiles in Norway 2002. Source: Statistics Norway and own calculations.

with this panel is that the poorest households have negative net wealth and a whopping 310% leverage of total assets. Much of this is presumably due to low-income students with few real or financial assets but relatively large loans (which, fortunately, do not require collateral). The background numbers however, reveal that only about 23% of the first quartile's debt are student loans, implying 65% of total debt being unsecured. Although the numbers in the first column of Table 1.4 are negative for housing and stocks, it is not the case that the poorest households hold negative amounts of these assets (or positive amounts of the risk-free asset for that matter): since debt overall outweighs all other holdings, net wealth is negative and thus positive holdings appear with minus signs in the tabulation.

Further, it is clear that the portfolio compositions evolve much in the same manner as when we considered age groups: wealthier households hold less of their net wealth in housing and are less indebted. But, while older households generally hold a smaller share of stocks in their portfolio, wealthier households hold larger equity shares than poorer households.

Curiously, the fourth quartile's portfolio seems to match the portfolio of the 55-66 cohort almost perfectly. Both hold about 90% of their wealth in housing, owe roughly 2.3% of net worth, and own quite a lot of stocks. Moreover, the average fourth quartile household is relatively richer and has placed over 10% of its worth in risky assets, while the 55-66 cohort has settled on 6.4%. Of course, learning that people in their 50s and early 60s share behavioural traits with those in the richest quartile of the population is hardly surprising (as we would expect these groups to overlap considerably).

### 1.1.3 Homeownership rates

The portfolio tables reported average holdings of "housing," but did not say anything about how many households were actually homeowners. Table 1.5 however, shows that the Norwegian homeownership rate is generally high and that owner-occupier status tends to follow income and age (where again the former closely follow the latter in the aggregate). The first row tells us that most people enter the housing market early and remain owners throughout their lives. The decline in the ownership rate after age 80 is not surprising as many then

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(together with oh, so many others) slip through.

Cohort	20-29	30-39	40-49	50-66	67-79	80-
Owner-occupiers	0.69	0.78	0.86	0.89	0.86	0.72
Income quintile	1st	2nd	3rd	4th	5th	
Owner-occupiers	0.61	0.75	0.84	0.91	0.95	

Table 1.5: Homeownership rates by age and income. Norway 2001 and 1997. Note: Rates by age are from the Population and housing census 2001 (Statistics Norway) while rates by income are from Gulbrandsen (2004) and refers to a 1997 survey. These two measures are not directly comparable since the former asked how many who lived in an owner-occupied dwelling while the latter asked how many households owned their dwelling.

move into nursing homes, “retirement communities” (and, in principle, become renters), or perhaps low-maintenance rental units.

The second row confirms a growing suspicion: as people become wealthier and can afford a housing purchase, they prefer homeownership to renting. Almost all high-income households own their dwelling, while nearly 40% of the first quintile households do not. A relative increase in housing prices will thus result in an exacerbation of social inequality by redistributing wealth towards those who already are high-income earners and homeowners. Overall, the total homeownership rate in Norway is roughly 0.8, which is very high internationally (Gulbrandsen, 2004). The rate has been stable and high the last couple of decades.

## 1.2 Asset returns

In the next chapters we will try to calculate the optimal household portfolios by employing a static model and a dynamic model. The models will be formulated so that only three assets are available to the investor: housing, stocks, and a risk-free asset. We will therefore be needing some estimates of expected returns, variances, and covariances. Norwegian data from the period 1992-2006 will be used due to the availability of the house price index and because we want to be able to compare our results to the observed 2002 portfolios.<sup>9</sup> As the whole topic of accurate asset returns is really not the main focus of our discussion, we will allow ourselves to take a not-so thorough approach below. Rather than setting heaven and earth in motion searching for the best possible estimates, we will settle for what seems “reasonable.”

<sup>9</sup>The reason for starting in 1992 is that Statistics Norway did not report housing price changes until the launch of the house price index (HPI) in 1991. Indices have later been constructed going all the way back to the 1800s, but the HPI itself is quite new. We will discuss longer perspectives below, but for the main analysis we will stick to our 14 year horizon for which we have reliable and easily available data.



### 1.2.1 Risk-free return

Since we will assume that a single risk-free asset can be held in both positive (bank deposits or bonds) and negative (debt/mortgage) amounts it seems now reasonable to determine the rate of return,  $r_f$ , somewhere between the actual deposit- and mortgage rates. We will therefore use the average real bond rate as reported by Statistics Norway for this purpose.<sup>10</sup>

The assumption may be justified as follows: Households with significant holdings of the risk-free asset are expected to seek the best possible interest on their “investment” given that there is no risk involved. Would they not place at least some of their money in long-term bonds, “safe” market funds, and in dedicated high-interest accounts? That is, assuming that households with positive net holdings of the risk-free asset only receive the regular bank deposit rate does not sound intuitively right; they should be able to do better.

Borrowing households obviously can not do much better than the bank’s regular mortgage rate, which lies fairly close to the bond rate for the period under discussion. When people decide to take up a mortgage they may elect to keep the rate fixed for some years (perhaps even for the full mortgage term), but we will abstract from such a complication here. Looking at Figure 1.1 however, we notice that the bond rate is more stable than the bank rates, so again, the bond rate seems like a decent compromise when we know that borrowing households do have a choice between fixed and floating mortgage rates. It might however be worth mentioning that Norwegian borrowers almost always go for the floating rate while fixed rate mortgages are much more popular in the U.S. and in many other countries (van Hemert, 2006).

For our period, 1992-2006, we have used quarterly data and found that the annual mean real bond rate was 0.0413 with a standard deviation of 0.0145. This is as good as risk-free when we remember that a standard deviation of 0.0145 means that over two-thirds of the observations lie in the interval [0.027, 0.056] when the asset return is assumed to be normally distributed (a cornerstone of the next chapter’s basic portfolio model). None of the observations were negative. Figure 1.1 shows that there has been a clear decline in the rate since the early 1990’s, which was a period of financial distress in Norway and in many other countries.

To keep things simple, we will adjust our estimate of the rate of return for 28% taxation on capital gains and interest (Sørensen, 2005). This of course lowers the implied mortgage rate as well, which makes good sense since mortgage interest payments are tax deductible. The real, after-tax, risk-free return is thus approximated at  $0.0413 \cdot (1 - 0.28) \approx 0.03$ .

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<sup>10</sup>Mortgage-, deposit-, bond- and inflation rates are from Statistics Norway’s *Statistical yearbook 2007*.

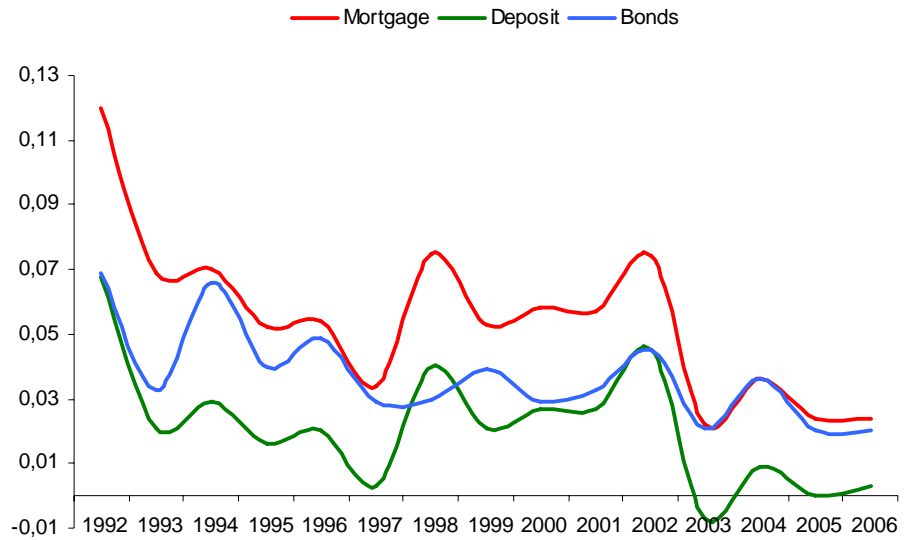


Figure 1.1: Real mortgage-, deposit- and bond rates. Smoothed annual observations. Source: *Statistical yearbook 2007*, Statistics Norway.

### 1.2.2 Stock market return

Holding stocks, or equity rather, yield two streams of return; capital gains<sup>11</sup> and dividends. These streams are taxed differently in different tax regimes, but usually at the same rate. We will use the development on the Oslo Stock Exchange over the 1992-2006 period to find numbers on the risk and return from what we will generically call “stocks.” We will hereafter assume that these are units/shares of well-diversified stock market portfolios. In Norway, the statutory tax rate on the capital gain component has been (and still is) 28% since 1992, while dividends was not taxed at all until 2005.<sup>12</sup> In order to calculate the true after-tax return on stocks we thus need separate time series of prices and dividends. For the former we will use the OSEBXPR index which reports the evolution of stock prices at the Oslo Stock Exchange.<sup>13</sup> Since this index was not initiated until 1995 we use data from a historical price index<sup>14</sup> for the years 1992-1994. Statistics Norway’s macro model of the Norwegian economy, MODAG, includes a variable called RENAMF300,<sup>15</sup> reporting the dividend rate of return of Norwegian stocks, which we will use to find the total stock market

<sup>11</sup>If we’re lucky that is, capital loss might just as well be the case.

<sup>12</sup>The actual tax rate on capital gains is less than 28% due to significant allowances for retained profits, but we will abstract from this complication here (Sørensen, 2005).

<sup>13</sup>Time series provided by Oslo Børs Informasjon AS.

<sup>14</sup>Klovland (2004b), more on this in the next section.

<sup>15</sup>Thanks to Statistics Norway for providing the time series!

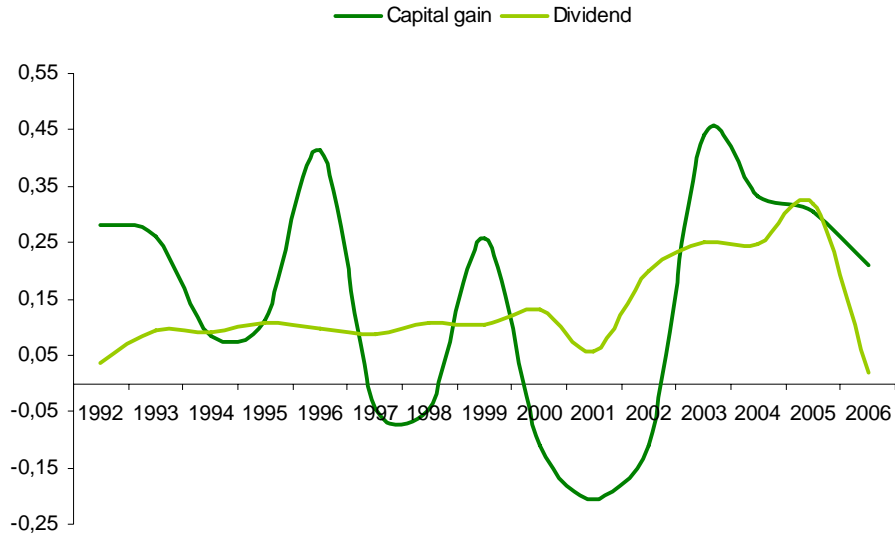


Figure 1.2: After-tax capital gain rate and dividend rate. Smoothed annual observations. Source: Oslo Stock Exchange, Norges Bank and Statistics Norway.

return.

Due to availability of data, we use annual series for the expected return and find that the mean after-tax capital return rate was 0.105 for 1992-2006. The average (non-taxed) dividend rate was 0.13 in the same period, which sounds very high. From Figure 1.2 we see a tremendous rise in dividends right before 2005, the year the dividend tax was introduced. Apart from this peak, however, it is clear that the rate was fairly stable in the 1990s and early 2000s with an average of 0.104 for 1992-2002. We will therefore use the latter as our estimate of the dividend rate and consider the observations of 2003-2005 an abnormality.<sup>16</sup>

Summing the two return components and adjusting for inflation yield an annual mean real rate of return of 0.188. The volatility of stock market returns is, for computational convenience, measured by the regular benchmark index<sup>17</sup> (since the tax rate is constant) and inflation. Now using quarterly series<sup>17</sup>, we find the standard deviation of the 1992-2006 sample to be roughly 0.2.

<sup>16</sup>Larger than usual dividends were paid out due to the imminent introduction of dividend taxation.

<sup>17</sup>OSEBX, available from the Oslo Stock Exchange and from Statistics Norway.

### 1.2.3 Housing as an asset and a consumption vehicle

It is far from obvious how to estimate the risk and return of housing, our third type of asset. Similar to equity, housing yields two streams of return: capital gain or loss, and the benefit from living in one's owned dwelling. We will measure the capital gain component by the house price index (HPI) as it is reported by Statistics Norway. The monthly 1992-2006 series was first transformed<sup>18</sup> into a quarterly series which was then used to find the quarterly returns of the capital component. Similarly constructed quarterly inflation rates were used to find the real return.

#### Housing dividend

The housing service received from a self-owned dwelling may be called "dividend" but is unlike its equity counterpart in that it is not measured in money, but rather in units of utility. The latter is naturally not observable and thus our first challenge is to find some approximating measure so that we may estimate the true total return from the housing asset.

The rationale for including the dividend is clear: if the household for some reason could not enjoy the fruits of homeownership (i.e., not been able to live in one's dwelling) it would have had to buy housing services somewhere else at some cost. But, since owner-occupiers do consume (or lease out) the housing service emitted from their dwelling, they implicitly receive the equivalent of this cost in what we will call dividend or imputed rent. At first the computation of this imputed rent may seem straightforward: just use data on actual rents and adjust for dwelling size. However, there are a number of problems with this approach.

First, self-owned dwellings and leased dwellings are typically very different qualitatively. Rented units are most often found in urban areas and in apartment buildings while self-owned units are larger and more often in rural and suburban areas. In fact, rented dwellings are on average only half the size of self-owned dwellings.<sup>19</sup> Would it then be reasonable to assume that the rent-to-house value ratio of the former is representative for that of the latter? Not necessarily.

Second, there may be a significant element of moral hazard in the rental market. It is often assumed that renters have weaker incentives to take sufficient "care" of their dwelling than owners (Brueckner, 1997), so that rents should be somewhat inflated due to landlords rolling the added depreciation over on the tenants' rents. Of course, this element would in principle just add to the implied dividends as well if the household considers renting and owning perfect substitutes.

Third, people's taste for homeownership is not entirely financial as they might experience substantial additional "utility" from owning rather than renting even if the dwelling is qualitatively the same. Less restrictions on own

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<sup>18</sup>Three-month average as quarterly observation.

<sup>19</sup>*Rental market survey 2005*, Statistics Norway.

behaviour and dwelling customisation are obvious examples. Such an argument implies a higher dividend than rent for an identical housing unit.

Fourth, as we discuss briefly below, rents are subject to taxation while housing dividends are not. That is, renting is less efficient and less attractive for all parties involved.

Despite all objections, by far the most realistic estimator of housing dividend must be rents of a similar (also geographically) dwelling, although perhaps adjusted in some fashion. Trouble is, actual rent-to-house value data on owner-occupier representative dwellings are hard to come by. Ideally, the dividend should be estimated by a careful study of “owner-like” rental units and their rent-to-house value ratio. As a potential “second best” solution, we can find the rent-to-price ratio for the housing market as a whole. By dividing annual average rent per square metre by average price per square meter, we find an average ratio of 0.068 over 1992-2006.<sup>20</sup> To use this as an estimate of housing dividend we will, for one thing, have to subtract the depreciation rate, but we are not quite ready to settle with this. Let us discuss some other strategies first.

### Different approaches

Himmelberg *et al.* (2005) employ the user cost approach of Poterba (1984) and find that the dividend-to-house value ratio should be roughly 0.05 under a set of reasonable assumptions.<sup>21</sup> That is, the annual value of the housing service received is 5% of the dwelling’s market value, or in other words; a house’s market value should be around 20 times its annual “rent value.” As Himmelberg *et al.* point out however, rents have not kept up with prices since the late 1990s.

Harding *et al.* (2004) on the other hand, divide the aggregate rents paid on the total stock of leased units. This produces the desired ratio, but it may be flawed due to inaccurate tax reporting (as data originate from filed tax reports via the national accounts) and the problem of representativeness discussed above. That said, it does sound like a fair estimate of housing dividend from the population of rental units at least, and it may perhaps be the closest we get without submerging ourselves in a detailed analysis of said topic. For 1992-2002 they settle on an estimate of 0.04 of the housing dividend.

Røed Larsen and Weum (2007) follow Case and Shiller (1989) in assuming that the average dividend-to-house value ratio is 0.05.<sup>22</sup> They further use repeat sales data from OBOS, a housing co-op in Oslo, and a rental index to compute

<sup>20</sup> Average annual rents per square metre was found for 2006 from the *Rental Market Survey 2007* and calculated backwards using the rent component of the consumer price index. Online: [http://www.ssb.no/english/subjects/08/02/30/lmu\\_en/](http://www.ssb.no/english/subjects/08/02/30/lmu_en/)

<sup>21</sup> Sadly, with our data on house price appreciation, the user cost measure is not at all to our liking...

<sup>22</sup> The 0.05 ratio assumption is very common in the literature: it may seem as if this should be true in a well-adjusted market with no price bubbles or any other imbalances. Aaron (1970) test 0.04 and 0.06 and finds the former most appropriate for his dataset. Jud and Winkler (2005) report the average imputed rent (1978-2001) in the U.S. market to be around 0.058 and fairly stable over time (Bureau of Economic Analysis).

the return from housing. Since OBOS units are very similar to rental units (smaller, urban apartments), many of the before mentioned difficulties with using rental data to find imputed rents of self-owned dwellings are avoided, but the estimates may still not be representative for the typical self-owned dwelling. Relying on a rental index, they find that for OBOS units in Oslo, the dividend-to-house price ratio has gone from approximately 0.09 in 1992 to 0.03 a decade later. The key here is not necessarily these levels, since the period mean is essentially a guess, but that there has been a significant decline in the 1990s.

Flavin and Yamashita (2002) do not find their estimates from the rental market directly. Instead, they use American PSID data<sup>23</sup> and assume that the rental rate of any unit can be modelled as  $D_t = (r_f + \delta)P_{t-1} + \omega_t$  where  $r_f$  is the assumed constant real risk-free rate,  $\delta$  is the depreciation rate,  $P_{t-1}$  is house market value, and  $\omega$  is property tax. That is, if the dwelling was to be leased out, the owner would require just enough in rent to match the return otherwise yielded by the risk-free asset. Further, the owner has running costs of  $C_t = \delta P_{t-1} + (1 - \tau)\omega_t$  where  $\tau$  is the marginal income tax rate (reflecting the tax deduction from homeownership). The total return is house value appreciation plus dividend ( $D_t$ ), less costs of owning ( $C_t$ ):

$$\begin{aligned} r_{h,t} &= \frac{P_t - P_{t-1} + D_t - C_t}{P_{t-1}} \\ &= \frac{P_t + r_f P_{t-1} + \tau \omega_t - P_{t-1}}{P_{t-1}}. \end{aligned}$$

For Norwegian conditions, the property tax can for all practical purposes be set to zero<sup>24</sup> so that Flavin and Yamashita's expression is reduced to

$$r_{h,t} = \frac{P_t - P_{t-1}}{P_{t-1}} + r_f.$$

That is, dividend is just the real interest rate. This assumption is similar to that of van Hemert (2006) who sets the imputed rent equal to the mean real interest rate in his study of adjustable versus fixed mortgage rates. It is tempting to adopt this approach due to its simplicity and plausability.

Figure 1.3 plots the Norwegian real interest rate together with the rent-to-price ratio discussed above, both normalised with respect to their period mean. The connection is clear and it seems very reasonable to use the interest rate as a dividend estimate. The former was found above to be 0.03 but let us add a percentage point to reflect the benefits from owning relative to renting, so that we get a rate of 0.04. This also fits well with the rent-to-price ratio when we

<sup>23</sup> *Panel Study of Income Dynamics*, 1968-1992. Homeowners were asked how much their house would sell for "today."

<sup>24</sup> In Norway, the question of whether or not to impose property tax is the prerogative of local authorities. Although roughly 50% of the population live in municipalities where property taxes are collected, the tax itself is very modest at a maximum rate of 0.7% of taxable property value. And, as mentioned earlier, taxable value is found to be roughly 20% of actual market value, according to Statistics Norway.

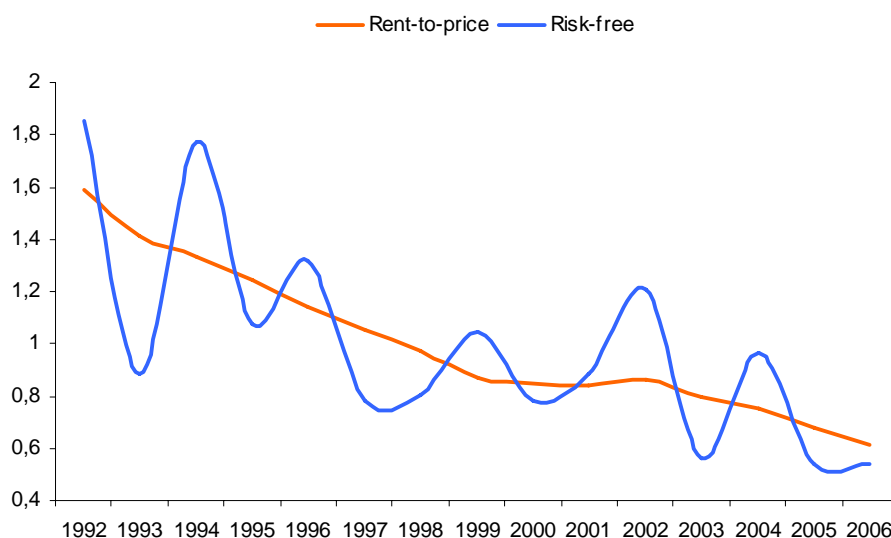


Figure 1.3: Normalised rent-to-price ratio and “risk-free” real interest rate. Smoothed annual observations. Source: Statistics Norway and own calculations.

subtract a depreciation rate of 0.03.<sup>25</sup>

### Total return to housing

The mean, annual, real housing price index growth rate was 0.071 over 1992–2006 with a standard deviation of 0.056. With the above assumed dividend rate of 0.04, this produces a total annual real return of 0.11.

#### 1.2.4 Housing risk

After making the simplifying assumption that dividends are as good as equal to the real return on bonds, the first bridge on the way to a description of the housing asset has been crossed. The second bridge crossing, which we will worry about now, requires decent estimates of the housing asset’s own variance and the covariances with the other two assets.

What we have basically said so far is that the return from a housing investment is the sum of the random variable measuring capital gain (HPI growth) and the constructed dividend rate. While the latter is almost a constant in the

<sup>25</sup>Statistics Norway reported a depreciation rate of roughly 0.03 for the housing stock in 2004 (see Chapter 3).

short run and should not be considered a random variable, the former inhabits significant volatility. Together however, the variance of this return series is way too low if we want to consider a single housing asset. Much of the risk associated with a single house is in fact eliminated by the aggregation of sales data regionally and nationally.

As the typical housing holdings of a household are its primary residence and maybe a holiday home, it is exposed to significant idiosyncratic risk and uncertainty. Not only with regards to future price development, but also with the quality of the stream of housing service. True, people are often insured against value (and service) decreasing contingencies such as water damage, construction error, fire, termite attack, etc. But what about a noisy new road practically next door, a new smelly smokestack on a nearby factory, “unfair” government expropriation, or simply a pair of bad neighbours? One may of course seek satisfaction by taking legal action against the wrongdoers, but this may be very costly.

On the other hand, homeowners may enjoy unanticipated excess *increases* in value and service as well. Think of a road (a bit away from the property, mind) greatly reducing the travel time into the city, (unexpected) permission to cut down trees hindering a great view, abolition of residential obligation,<sup>26</sup> or a general appreciation of one’s neighbourhood relative to other areas.

Case and Shiller (1989) mention that the annual standard deviation of return from a single house or property should be close to that of individual stock returns, in their case 0.15. Flavin and Yamashita (2002) use data from individual homeowner assessments (the PSID survey) and report a standard deviation of 0.14 for the period 1968-1992, while the stock market return standard deviation was 0.24 for the same sample. Goetzmann (1993) uses the same data as Case and Shiller and finds that the standard deviation of single house investment returns lies between 0.095 and 0.127 for four American cities (1971-1985) by adding a disturbance term to the asset return. Standard deviation of stock returns was found to be 0.185.

If we look at the ratio of single house standard deviation-to-stock market standard deviation in the latter two papers, we find this to be 0.58 and 0.6 respectively.<sup>27</sup> Englund *et al.* (2002) use different investment horizons, but we find the ten quarter horizon ratio to be 0.54.

We will assume a 0.6 ratio to be a decent approximation and thus set the standard deviation of our housing return to 0.12 (since we have found the stock market standard deviation to be roughly 0.2). van Hemert (2006) took a similar approach by fixing the housing standard deviation at 0.15 after Case and Shiller. But, simply scaling the housing return around the its mean accordingly (as in van Hemert’s paper) will not do in our case as we are also looking for covariances and correlations with other asset returns. The key property of the idiosyncratic risk component is of course that it is assumed uncorrelated with everything else.

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<sup>26</sup>Especially relevant for potential holiday properties in attractive areas, e.g. the skerries of Southern Norway.

<sup>27</sup>Average standard deviation of the four cities for the Goetzmann paper.



	<i>Housing</i>	<i>Stocks</i>	<i>Risk-free</i>
<b>Mean</b>	0.110	0.188	0.030
<b>SD</b>	0.120	0.200	0.011
<b>Variance and covariance</b>			
<i>Housing</i>	0.016		
<i>Stocks</i>	0.008	0.039	
<i>Risk-free</i>	-0.000	0.000	0.000
<b>Correlation</b>			
<i>Housing</i>	1.000		
<i>Stocks</i>	0.332	1.000	
<i>Risk-free</i>	-0.027	0.032	1.000

Table 1.6: Asset return estimates based on actual asset returns over 1992-2006. Source: Statistics Norway, Norges Bank, Oslo Stock Exchange and own calculations.

We may write the housing return as

$$r_{h,t} = r_{HPI,t} + r_{DIV,t} + \varepsilon_t$$

where  $\varepsilon_t$  is an independently normally distributed disturbance term with zero mean and a standard deviation such that the total asset return's standard deviation goes asymptotically towards 0.12 (annual standard deviation of the capital gain plus dividend time series was found to be 0.055 meaning that the disturbance term must contribute with 0.065). We use the random number generator with a normal distribution in Microsoft Excel to find the disturbance time series which we subsequently add to the original housing series. Of course, with only 60 realisations, the new housing series' standard deviation may not be *exactly* 0.12, but we pick the random number draw that has a mean closest to zero.

### 1.2.5 Estimates

From our three asset return series; housing, stocks, and risk-free, we compute the covariances and correlations between them in Excel. The results, together with mean returns and standard deviations, are summarised in Table 1.6.<sup>28</sup> Note that these are obviously not forecasts or “proper” econometric estimates, the latter would require much more sophisticated tools than those employed here, but they should be fairly representative for the 1992-2006 period.

Our numbers are almost in perfect harmony with Harding *et al.*'s (2004) corresponding table. This is of course not very surprising given that we are considering the same economy over a similar time period. The only notable

<sup>28</sup>The standard deviation of the risk-free asset is of course zero by assumption, but we are including the observed figure here for comparison's sake.

difference is their higher estimate on stock market standard deviation (0.31). Flavin and Yamashita (2002), Englund *et al.* (2002) and Goetzmann (1993) however, all report negative or zero correlation between single housing return and stock market return. This stands in sharp contrast to our result of a significant 0.33. Further, Flavin and Yamashita's mean returns on housing and stocks for 1968-1992 are much lower than ours; 0.066 and 0.082 respectively. Without trying to provide a proper explanation for this seeming inconsistency, we may note that all three papers above consider time periods ending roughly as ours start, the early 1990s, thus not picking up the recent boom in both real estate and stock prices.<sup>29</sup> We will see below that the period we have chosen is in fact extraordinary in a longer perspective.

Since we have included a random disturbance term in the housing series we might suspect this component to be contributing to the stock market correlation. After all, stock market return fluctuations are not all that different from the movements of such a disturbance variable.<sup>30</sup> We find however, that the correlation without adding the disturbance to the housing series is 0.43. Hence, our method for incorporating idiosyncratic risk has reduced the housing return's correlation with other asset returns (as expected and intended).

### Historical perspective

We briefly discussed the development of prices on stocks and housing in the Introduction and it became clear that the years since 1992 have mostly been a period of tremendous growth in both markets. Hence, the 1992-2006 period should produce unusually high return estimates for stocks and housing. We will therefore conduct a somewhat informal investigation of the time before and after 1992 using historical data on housing- and stock prices, and on historical real bond rates.<sup>31</sup> For 1920-2006<sup>32</sup> we calculate real annual (assumed risk-free) bond rates and house- and stock price growth rates. Note that we only consider *prices* for housing and stocks since we do not have stock market dividend data for the long period.

For the risk-free asset we find that the before-tax real return was roughly 2.3% for 1920-1991 against 3.5% for 1992-2006.<sup>33</sup> As the standard deviation was 6.2% for the former period we can safely say it was not very risk-free, but, as expected, most of this can be attributed to inflation volatility: inflation's standard deviation was actually slightly higher at 6.23%.

<sup>29</sup>Himmelberg *et al.* (2004) and Shiller (2005).

<sup>30</sup>In fact, we will *assume* below that the returns are indeed normally distributed!

<sup>31</sup>Annual time series collected from Klovland (2004a), Klovland (2004b), and Eitrheim and Erlandsen (2004). Inflation series from Grytten (2004).

<sup>32</sup>We choose 1920 since this is the first year of the bond series. For housing- and stock prices we have data from 1819 and 1914 respectively, but we will stick to 1920 here as well.

<sup>33</sup>This is a bit lower than what we found above when we did it "properly," and the discrepancy is presumably due to the coarser historical data and to different types of bonds being included each aggregate.

	<i>Housing</i>	<i>Stocks</i>	<i>Risk-free</i>	<i>Inflation</i>
<b><math>\Delta</math> mean</b>	0.081	0.133	0.012	-0.014
<b><math>\Delta</math> SD</b>	-0.053	0.038	-0.051	-0.061
<b><math>\Delta</math> correlation</b>				
<i>Housing</i>	1.000			
<i>Stocks</i>	0.386	1.000		
<i>Risk-free</i>	-0.557	-0.631	1.000	
<i>Inflation</i>	-0.552	-0.614	-0.089	1.000

Table 1.7: Difference between periods 1920-1991 and 1992-2006. Note: Prices only for housing and stocks. The values in the correlation matrix are in nominal terms. Source: Statistics Norway, Norges Bank, Oslo Stock Exchange and own calculations.

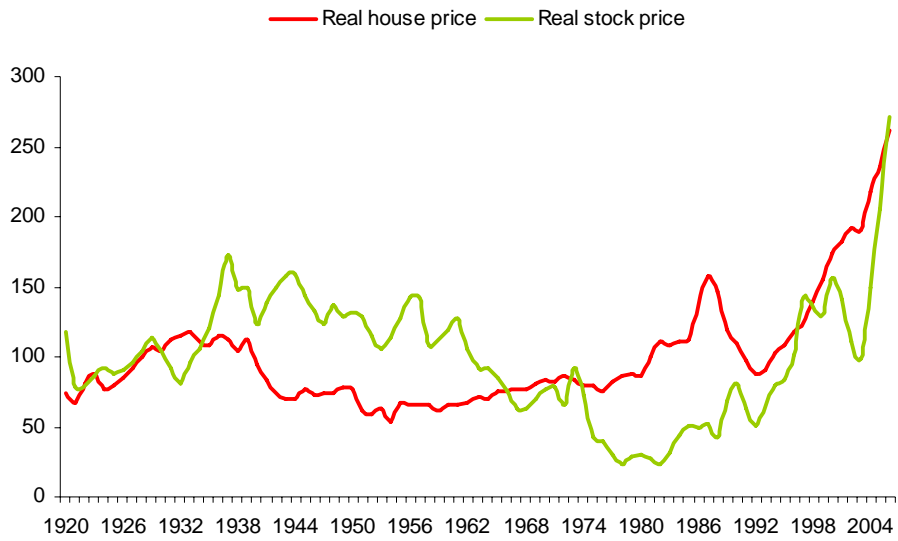


Figure 1.4: Real housing- and stock price indices normalised with 100 as long-run mean. Smoothed annual observations. Source: Statistics Norway and own calculations.

Housing- and stock prices both show surprisingly low real growth in the long-run: 0.6% and 0.2% respectively for the 1920-1991 period. Figure 1.4 plots the real, normalised price level of housing and stocks, and it shows clearly how strong the growth has been since the mid-1990s. From our price data we find mean real capital returns to be 8.7% and 13.6% for housing and stocks<sup>34</sup> over 1992-2006, and we can thus conclude that the previous section's estimates are not at all in line with historical long-run averages.

Table 1.7 shows the difference between the 1992-2006 and 1920-1991 periods in terms of means, standard deviations, and correlations. The latter is particularly interesting since we ran into some differences with prominent papers above concerning the correlation between housing- and stock market return. We now find that the two prices are vastly more correlated in the later period than in the long-run: for 1920-1991 the statistic is a modest 0.027 against 0.413 for 1992-2006. Apart from a stock market slump around 2003, we see from Figure 1.4 that both prices “took off” almost simultaneously in the mid-nineties. No wonder why newer correlation estimates are higher!

Moreover, the correlations in Table 1.7 are for nominal series so that we can control for the impact of inflation. We see that the correlations between real housing- and stock prices and the risk-free rate were much stronger in the long period due mostly to inflation movements. Background data shows that real housing- and stock price growth had correlations with the real risk-free rate of 0.41 and 0.11 respectively in the long period, and 0.23 and -0.1 in the 1992-2006 period.

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<sup>34</sup>As stressed earlier, these figures cannot be compared to those we found in the previous section as we are now using different data sets, we are not including dividends, and we are not adjusting for taxes.

## Chapter 2

# Static portfolio model

The discussion in this chapter will make use of a central assumption in financial economics, namely that an investor composes a portfolio of assets based only on the latters' expected returns and expected variance. It is further assumed that the investor is risk averse so that higher risk (in the form of greater portfolio variance) must be compensated with higher expected return. Further, the model presented in the first section is a *static* model. This means that the investor is assumed to treat all periods the same such that a general optimisation problem is reduced to a study of the single period problem. This might not be very realistic but it can nevertheless deliver some valuable insights into households' behaviour.

### 2.1 A simple mean-variance framework

We will use the simplest possible framework in our analysis of the household asset allocation decision. In setting up a very basic mean-variance finance model for portfolio selection, we will follow Danthine and Donaldson's (2005) textbook and their approach. Let us first assume that only two risky assets, housing and stocks, are available for the household. The asset returns,  $r_h$  and  $r_s$ , are modelled as random variables with some expected value and some variance. Any combination of these two will be called the *risky* portfolio, and has a total return of

$$r_p = wr_h + (1 - w)r_s,$$

where  $w$  is the portfolio weight on the housing asset. We write the first two moments of the portfolio return as

$$\begin{aligned} E(r_p) &\equiv \mu_p = w\mu_h + (1 - w)\mu_s, \\ Var(r_p) &\equiv \sigma_p^2 = w^2\sigma_h^2 + (1 - w)^2\sigma_s^2 + 2w(1 - w)\sigma_{hs}, \end{aligned}$$

using the standard rules. The expected return is thus the weighted average of the expected returns of the two assets, while the portfolio variance may be smaller

or greater than the weighted average depending on the two assets' covariance. If the latter is negative we see that the benefit from diversification is a reduction in portfolio variance (i.e., risk).

The investor household will seek the weight on the housing asset,  $w$ , that minimises the portfolio variance *given* some expected return.<sup>1</sup> Formally:

$$\begin{aligned} \min_w \{ & w^2\sigma_h^2 + (1-w)^2\sigma_s^2 + 2w(1-w)\sigma_{hs} \} \\ \text{s.t. } & w\mu_h + (1-w)\mu_s = \mu_p. \end{aligned}$$

That is, by fixing  $\mu_p$  at various values while minimising the variance, we obtain points in mean-variance space that trace out the frontier of feasible portfolios. The problem's constraint yields

$$w(\mu_h - \mu_s) = \mu_p - \mu_s \iff w = \frac{\mu_p - \mu_s}{\mu_h - \mu_s}$$

which eliminates  $w$  from the portfolio variance such that

$$\sigma_p^2 = \frac{(\mu_p - \mu_s)^2}{(\mu_h - \mu_s)^2}(\sigma_h^2 + \sigma_s^2 - 2\sigma_{hs}) + \frac{\mu_p - \mu_s}{\mu_h - \mu_s}(\sigma_{hs} - \sigma_s^2)2 + \sigma_s^2.$$

To find the shape of the parabola (or the hyperbola)<sup>2</sup> of risky portfolios it will be useful to rearrange this into a clearer function of  $\mu_p$ , the portfolio return, and some constants. It is shown in Appendix A that we can write

$$\sigma_p^2(\mu_p) = A\mu_p^2 + B\mu_p + C \quad \text{or} \quad \sigma_p(\mu_p) = \sqrt{A\mu_p^2 + B\mu_p + C}, \quad (2.1)$$

where

$$\begin{aligned} A &= \frac{\sigma_h^2 + \sigma_s^2 - 2\sigma_{hs}}{(\mu_h - \mu_s)^2} \\ B &= \frac{-2\mu_s\sigma_h^2 - 2\mu_h\sigma_s^2 + 2\sigma_{hs}(\mu_h + \mu_s)}{(\mu_h - \mu_s)^2} \\ C &= \frac{\mu_s^2\sigma_h^2 + \mu_h^2\sigma_s^2 - 2\mu_h\mu_s\sigma_{hs}}{(\mu_h - \mu_s)^2} \end{aligned}$$

are constants.

The slope of the hyperbola is then easily found to be

$$\frac{d\sigma_p}{d\mu_p} = \frac{2A\mu_p + B}{2\sqrt{A\mu_p^2 + B\mu_p + C}}$$

by differentiating Equation (2.1).

---

<sup>1</sup>Alternatively, the household would seek the weight that *maximised* the expected return given some portfolio variance. These two formulations are equivalent since they both define the trade-off between return and risk.

<sup>2</sup>The bullet-shaped portfolio frontier in the  $(\mu, \sigma^2)$  plane (or, the curve representing available portfolios for different values of expected return and variance) is called a *parabola*. When we consider the  $(\mu, \sigma)$  plane, it is called a *hyperbola*.

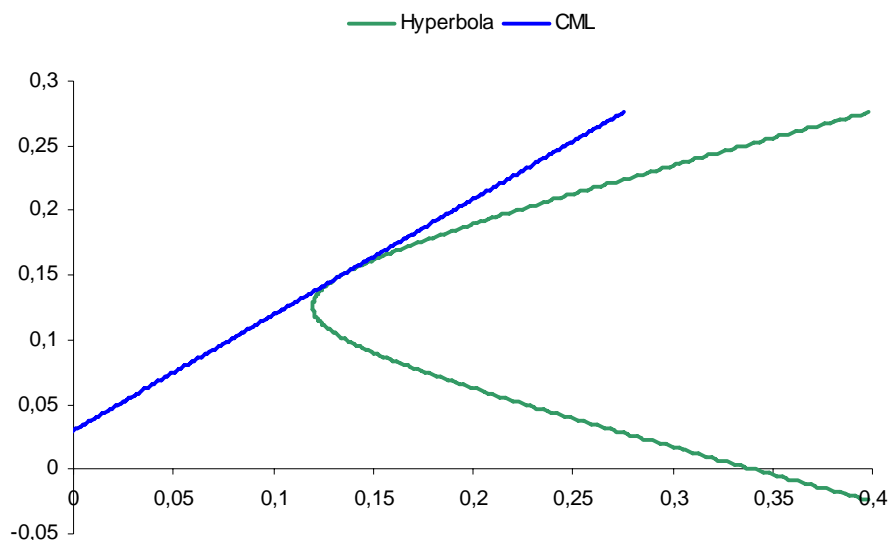


Figure 2.1: Hyperbola of portfolios and the capital market line as produced by the estimates in Table 1.6. Expected return on the vertical axis, standard deviation on the horizontal axis.

### 2.1.1 The capital market line

Next, we introduce a risk-free asset with predetermined return  $r_f$ . The holdings are here allowed to be both negative and positive.<sup>3</sup> The portfolio frontier will now be a straight line from  $(0, r_f)$  through  $(\sigma_T, \mu_T)$  in the  $(\sigma, \mu)$  plane, called *the capital market line* (CML). As we see from Figure 2.1,  $(\sigma_T, \mu_T)$  is the tangency point between the straight line beaming from  $(0, r_f)$  and the bullet-shaped hyperbola, and we may refer to it as *the market portfolio*. The difference quotient of the CML is<sup>4</sup>

$$\frac{\sigma_T}{\mu_T - r_f}$$

which, with tangency, must equal the slope of the hyperbola such that

$$\frac{\sigma_T}{\mu_T - r_f} = \frac{2A\mu_T + B}{2\sqrt{A\mu_T^2 + B\mu_T + C}}. \quad (2.2)$$

<sup>3</sup>As we have not imposed any non-negativity constraints on the portfolio weights of housing and stocks,  $w$  and  $1 - w$ , we are actually allowing holdings of these assets to be negative as well. This is clearly not a reasonable assumption for the housing asset, but it is necessary if we want to find an analytical result. Fortunately though, the optimal holdings of housing (and stocks) turn out to be positive in our calculations below. Otherwise we would have had to resort to a numerical solution with the non-negativity constraints imposed.

<sup>4</sup>Danthine and Donaldson (2005:122).

Since Equation (2.1) holds for any point on the hyperbola, we can write

$$\sigma_T(\mu_T) = \sqrt{A\mu_T^2 + B\mu_T + C}. \quad (2.3)$$

Together, (2.2) and (2.3) determines the tangency point  $(\sigma_T, \mu_T)$  and thus the portfolio frontier given  $r_f$ . Equation (2.3) inserted in (2.2) yields<sup>5</sup>

$$\mu_T = -\frac{2C + Br_f}{B + 2Ar_f} > 0, \quad (2.4)$$

which we will employ below to find the actual tangency point using our data and estimates. Note now that we are able to locate the market portfolio without knowing the individual household's preferences. The *two-fund separation theorem* (Danthine and Donaldson, 2005) ensures that any investor, regardless of risk preference, will choose a portfolio that is a combination of the risky market portfolio  $(\sigma_T, \mu_T)$  and the risk-free asset. The individual household's degree of risk aversion will thus determine the proportion of risky assets in the total portfolio.

### 2.1.2 Household preferences

In order for us to determine the household's preferred portfolio in the framework above, preferences must be represented by a function of any portfolio's expected return and standard deviation. To achieve this we must either assume a quadratic utility function or that asset returns are normally distributed. The former assumption is undesirable because it implies putting a restriction on the portfolio return to avoid negative marginal utility, and because it yields an increasing coefficient of absolute risk aversion.<sup>6</sup> The normality assumption is not consistent with limited liability since it may very well predict a return below negative 100%. But, if we instead assume that it is the continuously compounded returns that are normally distributed, the problem is evaded.

Before we discuss this any further, suppose the household's preferences can be represented by the *constant relative risk aversion*, CRRA, function:<sup>7</sup>

$$u(r_p) = \frac{(1 + r_p)^{1-\gamma}}{1-\gamma} \quad (2.5)$$

where  $\gamma$  is the Arrow-Pratt measure of relative risk aversion. Equation (2.5) implies constant relative risk aversion  $\gamma$ , meaning that the household is assumed to invest the same proportion of total wealth in risky assets regardless of its

<sup>5</sup>Intermediate calculations are available in Appendix A.

<sup>6</sup>This is not realistic, *decreasing* absolute risk aversion is actually more consistent with data.

<sup>7</sup>The treatment of the CRRA utility function in this section follows to a considerable extent Söderlind's (2007) lecture note on finance theory.



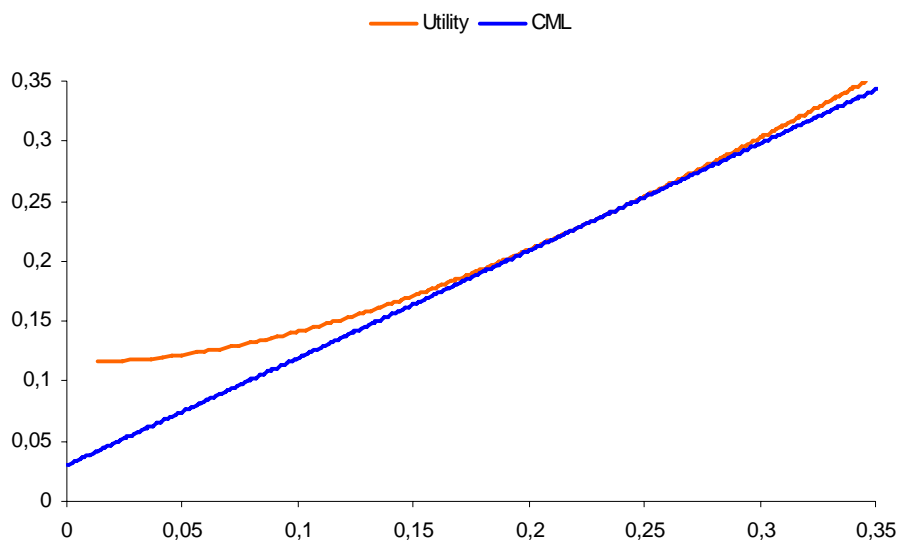


Figure 2.2: Household indifference curve and the capital market line as produced by the estimates in Table 1.6. Expected return on the vertical axis, standard deviation on the horizontal axis.

actual level of wealth. This is a fairly reasonable assumption as long as the changes in wealth do not become too large (Varian, 1992).<sup>8</sup>

The continuously compounded portfolio return is written  $\hat{r}_p = \ln(1 + r_p)$  so that  $1 + r_p = \exp(\hat{r}_p)$ . Equation (2.5) then becomes

$$u(\hat{r}_p) = \frac{e^{(1-\gamma)\hat{r}_p}}{1-\gamma}. \quad (2.6)$$

Of course, the assumption above implies that the asset returns are *log-normally*<sup>9</sup> distributed, but the sum of two (or more) such random variables is *not* log-normally distributed. Log-normal distribution of the portfolio return should however, work just fine as an approximation to the true distribution

<sup>8</sup>We did see in Table 1.4 above that the share of risky stocks in the household portfolio remained unchanged from the second to the third wealth quartile. From the third to the fourth quartile however, the share of stocks rose over five times.

<sup>9</sup>A very informal test provides some justification for this assumption: histograms of real growth rates of the housing price (HPI) and the Oslo Stock Exchange benchmark index show distributions not that far from the typical log-normal, skewed pdf. Using the quarterly series<sup>7</sup> mentioned in Chapter 1 we find that both housing and stocks lean heavily to the right, but with significant negative observations. In fact, the stocks data is farther from a log-normal shape than the housing data. However, since the assumption of log-normally distributed stock market return is common in the literature, and since the housing returns seem to be even closer to the said distribution, we will maintain the above assumption in good conscience.

(whatever that might be), and it will allow us to use the CRRA utility function rather than a less realistic formulation.<sup>10</sup>

It can be shown that since  $\hat{r}_p \sim N(\hat{\mu}_p, \hat{\sigma}_p^2)$ ,  $E(e^{\hat{r}_p}) = e^{\hat{\mu}_p + \frac{1}{2}\hat{\sigma}_p^2}$  must hold (Söderlind, 2007). Applying this to, and taking the expectation of, Equation (2.6) yields

$$E[u(\hat{r}_p)] = \frac{1}{1-\gamma} e^{(1-\gamma)\hat{\mu}_p + \frac{1}{2}(1-\gamma)^2\hat{\sigma}_p^2}.$$

Next, perform the transformation

$$\frac{\ln(E[u(\hat{r}_p)] \cdot (1-\gamma))}{1-\gamma} = E[\hat{u}(\hat{r}_p)] = \hat{\mu}_p + \frac{1-\gamma}{2}\hat{\sigma}_p^2, \quad (2.7)$$

which is defined for  $E[u(\hat{r}_p)] < 0$  when  $\gamma > 1$ . We can then say that the right-hand side of Equation (2.7) represents the expected utility from the continuously compounded portfolio return. Next, when  $\hat{r}_p = \ln(1+r_p) \sim N(\hat{\mu}_p, \hat{\sigma}_p^2)$ , we must have

$$\begin{aligned} E(1+r_p) &= e^{\hat{\mu}_p + \frac{\hat{\sigma}_p^2}{2}}, \\ \ln E(1+r_p) &= \hat{\mu}_p + \frac{\hat{\sigma}_p^2}{2}, \\ \hat{\mu}_p &= \ln E(1+r_p) - \frac{\hat{\sigma}_p^2}{2}, \end{aligned} \quad (2.8)$$

and

$$\begin{aligned} sd(1+r_p) &= E(1+r_p) \cdot \sqrt{e^{\hat{\sigma}_p^2} - 1}, \\ var(1+r_p) &= [E(1+r_p)]^2 \cdot (e^{\hat{\sigma}_p^2} - 1), \end{aligned}$$

so that

$$\begin{aligned} \frac{var(1+r_p)}{[E(1+r_p)]^2} + 1 &= e^{\hat{\sigma}_p^2}, \\ \ln\left(\frac{var(1+r_p)}{[E(1+r_p)]^2} + 1\right) &= \hat{\sigma}_p^2. \end{aligned} \quad (2.9)$$

Inserting from Equations (2.8) and (2.9) in Equation (2.7) yields

$$\begin{aligned} E[v(r_p)] &= \ln E(1+r_p) - \frac{\hat{\sigma}_p^2}{2} + \frac{1-\gamma}{2}\hat{\sigma}_p^2 \\ &= \ln E(1+r_p) - \frac{\gamma}{2} \ln\left(\frac{var(1+r_p)}{[E(1+r_p)]^2} + 1\right) \\ &= \ln(1+\mu_p) - \frac{\gamma}{2} \ln\left(\frac{\sigma_p^2}{(1+\mu_p)^2} + 1\right) \end{aligned} \quad (2.10)$$

---

<sup>10</sup>Expected utility is actually not defined when we use the CRRA functional form together with normally distributed portfolio returns so we are forced to relax at least one of the desired assumptions.

	$\gamma = 5$	$\gamma = 6$	$\gamma = 7$	$\gamma = 8$	$\gamma = 10$	$\gamma = 12$	$\gamma = 14$	$\gamma = 16$
$\mu_c$	0.313	0.233	0.191	0.163	0.130	0.110	0.097	0.088
$\sigma_c$	0.317	0.227	0.180	0.149	0.112	0.090	0.075	0.065
$w_h$	1.177	0.844	0.670	0.553	0.416	0.333	0.279	0.241
$w_s$	1.195	0.857	0.680	0.562	0.422	0.338	0.283	0.245
$w_f$	-1.372	-0.702	-0.350	-0.115	0.162	0.329	0.438	0.514

Table 2.1: Expected portfolio return, standard deviation, and portfolio weights as produced by the estimates in Table 1.6.

Equation (2.10) is now a utility function in the mean-variance plane of the portfolio return, and will be used to determine optimal portfolio allocations for the household. Figure 2.2 shows the indifference curve for  $\gamma = 6$  together with the capital market line, based on the estimates in Table 1.6.

## 2.2 Optimal portfolio allocations

The estimates in Table 1.6 of Chapter 1 allow the computation of our *ABC* constants;

$$\begin{aligned} A &= 6.4103 \\ B &= -1.6154 \\ C &= 0.1161 \end{aligned}$$

which, inserted in Equation (2.4), produce  $\mu_T = 0.1493$  as well. With these four ingredients we find from Equation (2.3) that  $\sigma_T = 0.1335$ . As mentioned earlier, the slope of the tangent, and thus also the capital market line, is given by  $\sigma_T/(\mu_T - r_f)$  in the  $(\sigma, \mu)$  plane. Inverting this, we know that the capital market line must be, in the  $(\mu, \sigma)$  plane, given by

$$\begin{aligned} \mu_c &= r_f + \frac{\mu_T - r_f}{\sigma_T} \sigma_c \\ \mu_c &= 0.03 + 0.8936 \cdot \sigma_c. \end{aligned} \tag{2.11}$$

From Equation (2.11) we find the  $(\mu_c, \sigma_c)$  pairs that make up the CML for  $0.03 \leq \mu_c \leq 0.4$  in Microsoft Excel. To find the optimal points on this line according to the utility function in Equation (2.10), we search for the points giving the highest level of utility for various degrees of relative risk aversion (i.e., when the indifference curves are tangent to the capital market line, as in Figure 2.2). The results are reported in the upper half of Table 2.1.

Any portfolio on the CML will be a combination of the risky tangency portfolio  $(\mu_T, \sigma_T)$  and the risk-free asset. The weight on housing in the tangency

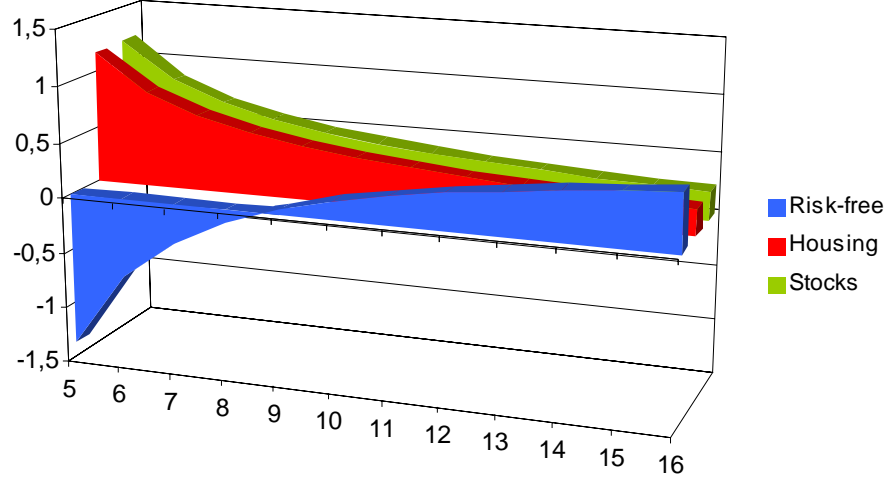


Figure 2.3: Optimal portfolio shares for different degrees of relative risk aversion with return estimates based on actual asset returns over 1992-2006.

portfolio follows from the constraint of the optimisation problem in the previous section,

$$\begin{aligned} w &= \frac{\mu_T - \mu_s}{\mu_h - \mu_s} \\ &= \frac{-0.0387}{-0.0780} = 0.4962. \end{aligned}$$

That is, the optimal portfolio should consist of slightly more stocks than housing. From Table 1.2 we know that this is a far cry from the observed holdings where the corresponding housing-to-stocks share was a whopping 0.93.<sup>11</sup> Even the households with the largest proportion of stocks in their portfolios, those in the fourth wealth quartile (Table 1.4), had a housing share as high as 0.89. It thus appears that people, given our return estimates, hold way too little equity relative to housing.<sup>12</sup>

Denote the overall weight of the risk-free asset by  $w_f$  so that the total portfolio weights on housing and stocks can be written  $w_h \equiv (1 - w_f)w$  and

<sup>11</sup>From Table 1.2:  $w = \frac{w_h}{w_h + w_s} = \frac{1.175}{1.264} = 0.93$ .

<sup>12</sup>Of course, as we mentioned in Section 1.1.1, stock holdings may have been unusually low around 2002 so the discrepancy might be somewhat exaggerated here.

$w_s \equiv (1 - w_f)(1 - w)$  respectively. From the *total* expected return we find

$$\begin{aligned}\mu_c &= w_h\mu_h + w_s\mu_s + w_fr_f \\ &= (1 - w_f)w\mu_h + (1 - w_f)(1 - w)\mu_s + w_fr_f \\ &= (1 - w_f) \cdot 0.1493 + w_f \cdot 0.03 \\ &= 0.1493 - w_f \cdot 0.1193,\end{aligned}$$

so that we obtain a weight on the risk-free asset for every point along the capital market line according to

$$w_f = \frac{0.1493 - \mu_c}{0.1193}.$$

The total weights on housing and stocks follow trivially and are presented in the lower half of Table 2.1, and illustrated in Figure 2.3. We see again that the fit with data is generally poor due to the housing- and stocks shares being virtually equal in the optimal portfolios. However, the optimal portfolios with lower relative risk aversion<sup>13</sup> tend to give a better description of young portfolios (Table 1.3), while higher risk aversion portfolios better fit older household observations. Further, wealthier households are by far the closest to “optimum” (Table 1.4) as they hold quite a lot of stocks relative to their other holdings.

### 2.2.1 Allocations with older estimates

It is clear that the static portfolio model is unable to explain the observed household behaviour given the return estimates found in Chapter 1, *alternatively* that households are dramatically underinvesting in the equity market. Regardless, we saw above that the 1992-2006 period was pretty unusual return-wise, and it should therefore be interesting to find the optimal portfolios given some expected returns<sup>14</sup> based on pre-1992 data as well. Perhaps most households’ beliefs about asset returns are lagged by a few years (or even decades)?

We mentioned in Chapter 1 that we have data (at least) all the way back to 1920 on housing- and stock prices, bond rates, and on inflation. But, since our data on stock market dividends only dates back to 1966, we will seek the optimal portfolio based on the asset returns over 1966-1991.

The annual mean stock price appreciation rate was 0.084 in this period, while the dividend rate was fairly stable at 0.065. Assuming a 0.28 tax rate on capital gains only, we find a nominal, annual rate of return of 0.125. What is possibly most remarkable about this period compared to 1992-2006, apart from the lower nominal returns, is the high rate of inflation. From Statistics Norway

<sup>13</sup>The lowest degree of relative risk aversion reported in Table 2.3, 5, is not really low, but the portfolios from degree 4 and lower are literally “off the chart” due to the slope of the capital market line.

<sup>14</sup>Expectations and beliefs about future returns are not really the topics here, but we have in a way implied that the 1992-2006 means are expectations formed around 2002 as they are in part based on then historical returns (1992-2001), and in part on future returns (2002-2006) which we must assume was expected to some extent.

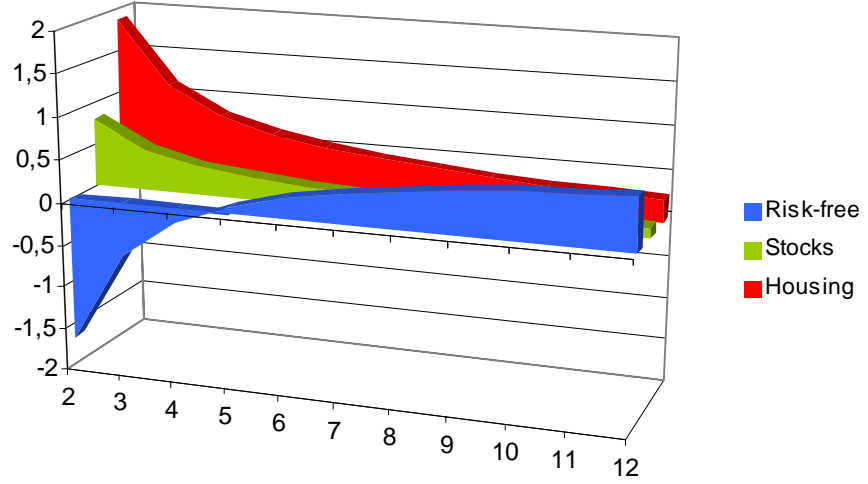


Figure 2.4: Optimal portfolio shares for different degrees of relative risk aversion with return estimates based on actual asset returns over 1966-1991.

data we find an annual mean inflation rate of 0.066. This yields a mean, real, after-tax rate of return of 0.06 for stocks. The standard deviation of this series was 0.18, slightly lower than for the 1992-2006 sample.

The housing price index grew on average at a nominal rate of 0.082 over the “historical” period, and, adjusted for inflation, we get 0.016 real growth per year. We will maintain the assumption from Chapter 1 of 4% real, annual housing dividends so that the total rate of return to housing can be set to 0.056. We will also assume that the housing risk-to-stock market risk ratio still is 0.6 so that the housing return has a standard deviation of 0.11, and further that the covariance between housing and stocks now is zero. The latter assumption provides a stronger incentive for diversification and is in line with the discussion at the end of Chapter 1.

For bonds we actually find a slightly negative real return over 1966-1991, but we will need this to be positive in our model so we “compromise” and set a real after-tax rate of 0.02 instead.

Going through exactly the same steps<sup>15</sup> as in we did above, we end up with the portfolio shares reported in Table 2.2 and in Figure 2.4 and with  $w = 0.7$ . The latter is much higher than in the “standard” case and it will presumably close some of the gap between observed and optimal holdings.

Going back to Table 1.4 we see that the holdings of the fourth quartile

<sup>15</sup>Details are available in Appendix A.

	$\gamma = 2$	$\gamma = 3$	$\gamma = 5$	$\gamma = 7$	$\gamma = 9$	$\gamma = 11$
$\mu_c$	0.120	0.080	0.054	0.044	0.038	0.035
$\sigma_c$	0.254	0.153	0.084	0.061	0.046	0.038
$w_h$	1.882	1.129	0.640	0.452	0.339	0.282
$w_s$	0.806	0.484	0.274	0.194	0.145	0.121
$w_f$	-1.688	-0.613	0.086	0.355	0.516	0.597

Table 2.2: Optimal portfolio shares from the older return estimates (1966-1991) listed in Table A.1 of Appendix A.

households match the optimal portfolio with a relative risk aversion degree of 5 quite well, even though the observed portfolios contain too little equity. Further, we see that the third and second quartile holdings fit better with the  $\gamma = 4$  and  $\gamma = 2$  portfolios respectively, and it certainly seems like we are on to something: richer households seem to be more risk averse than poorer households, and perhaps more importantly for us, the older return estimates (or beliefs) can explain a lot more than the newer figures.

The latter is possibly an indication that households' beliefs are outdated or somehow out of line with the true observed returns. We have seen that a lower real interest rate is a better match with people's actual behaviour: perhaps households have not fully accounted for the lower inflation in recent years? If households believe in a higher rate of inflation, they also believe in lower real interest rates and lower real returns on stocks and housing. We mentioned above that inflation has been, in a historical perspective, quite low since the early 1990s; maybe we can attribute some of the discrepancy between the model's predictions in the previous section and the observed behaviour to beliefs of higher inflation?

We have also seen (in Chapter 1) that the real price appreciation of stocks has been very low over the long-run: perhaps households underestimate the stock market return? After all, stock prices are the "visible" part of the stock return, while dividends may be more unknown (at least for those who have no experience from investing in the market).

When it comes to housing it is now important to remember that this type of asset is not just like any other financial asset: it is not liquid (so it is costly and even difficult to adjust) and holdings of it is much more likely to be driven by its consumption properties than its function as an asset. It is obviously unrealistic to assume that people constantly adjust their amount of living space according to expected price movements, but if they think (real) prices will only go up and up, they may very well upgrade or buy more at entrance.

## 2.2.2 Allocations with binding housing constraints

In Table 1.3 we found that the share of housing varies dramatically across cohorts, from 2.85 for the youngest to 0.74 for the oldest. Flavin and Yamashita

(2002) argue that the households' housing holdings are mainly determined by the housing consumption demand, and that the portfolio problem thus is *constrained* by this share. They go on to compute the optimal portfolios for each cohort given the corresponding housing share observed in the data. We will extend our model in the same spirit and try to find cohort portfolios when spending on housing is exogenously given.

First, if we assume  $w_h$  given and include the risk-free asset in the original problem formulation, we get

$$\begin{aligned} \min_{w_s, w_f} \{ & \bar{w}_h^2 \sigma_h^2 + w_s^2 \sigma_s^2 + w_f^2 \sigma_f^2 + 2\bar{w}_h w_s \sigma_{hs} + 2\bar{w}_h w_f \sigma_{hf} + 2w_s w_f \sigma_{sf} \} \\ \text{s.t. } & \bar{w}_h \mu_h + w_s \mu_s + w_f r_f = \mu_p. \end{aligned}$$

This can be simplified if we assume that the risk-free asset is not correlated with neither housing nor stocks<sup>16</sup> and that it is indeed risk-free (zero variance). Further, the weights must sum to one such that  $w_s = 1 - \bar{w}_h - w_f$ . We now have

$$\begin{aligned} \min_{w_f} \{ & \bar{w}_h^2 \sigma_h^2 + (1 - \bar{w}_h - w_f)^2 \sigma_s^2 + 2\bar{w}_h (1 - \bar{w}_h - w_f) \sigma_{hs} \} \\ \text{s.t. } & \bar{w}_h \mu_h + (1 - \bar{w}_h - w_f) \mu_s + w_f r_f = \mu_p. \end{aligned}$$

From the constraint we solve for the weight on the risk-free asset,

$$\begin{aligned} w_f (r_f - \mu_s) &= \mu_p - \bar{w}_h \mu_h - \mu_s + \bar{w}_h \mu_s, \\ w_f &= \frac{\mu_p + \bar{w}_h (\mu_s - \mu_h) - \mu_s}{r_f - \mu_s}, \end{aligned}$$

and insert the expression into the portfolio variance:

$$\begin{aligned} \sigma_p^2(\bar{w}_h, \mu_p) &= \bar{w}_h^2 \sigma_h^2 + \left( 1 - \bar{w}_h - \frac{\mu_p + \bar{w}_h (\mu_s - \mu_h) - \mu_s}{r_f - \mu_s} \right)^2 \sigma_s^2 \\ &\quad + 2\bar{w}_h \left( 1 - \bar{w}_h - \frac{\mu_p + \bar{w}_h (\mu_s - \mu_h) - \mu_s}{r_f - \mu_s} \right) \sigma_{hs}. \end{aligned}$$

We will now get a unique hyperbola for each  $\bar{w}_h$ , and, by searching for points on the hyperbolas that give the greatest utility for, say,  $\gamma = 6$ ,<sup>17</sup> we obtain optimal *constrained* portfolios for each cohort. The resulting portfolio weights are presented in the upper half of Table 2.3. The first thing we notice with this table is that households, given the amount held of housing, "should" be investing absurdly heavily in equity and borrow huge amounts of money for financing. Obviously, households will never be able to build up this kind of debt due to credit limitations and to the fact that equity cannot serve as collateral. For instance, it is not particularly plausible that the oldest cohort could (or should) have loans outweighing their housing assets over three times.

<sup>16</sup>This is very reasonable given our results from Table 1.5.

<sup>17</sup>We saw above that  $\gamma = 5$  gave the housing share closest to that in the data while  $\gamma = 7$  gave the closest debt share; it thus seems reasonable to run with  $\gamma = 6$ .



	-24	25-34	35-44	44-54	55-66	67-79	80-
$\bar{w}_h$	2.848	2.607	1.492	1.132	0.959	0.801	0.736
$w_s$	1.583	1.680	2.156	2.269	2.306	2.348	2.355
$w_f$	-3.431	-3.287	-2.648	-2.401	-2.265	-2.149	-2.355
$w_s^c$	1.000	1.000	0.953	0.819	0.780	0.797	0.830
$w_f^c$	-2.848	-2.607	-1.445	-0.951	-0.739	-0.598	-0.566

Table 2.3: Portfolio weights on stocks and risk-free given the weights on housing by cohorts. 1992-2006 return estimates. Note: The upper half of the table reports the unconstrained analytical solution, while the lower half reports the results from a numerical solution with credit constraints.

In order to get anywhere with this, we will have to impose some sort of *credit constraint* reflecting the fact that debt is limited (at least) by the holdings of real assets, say

$$w_f + (1 - \phi)w_h \geq 0,$$

where  $\phi$  represents the required mortgage prepayment. Unfortunately, we cannot incorporate this constraint analytically; we will have to resort to a numerical method. We will employ a simple procedure from Chen *et al.* (2008) and solve exactly the same problem as we laid out above, but now with  $w_f + w_h \geq 0$  imposed.<sup>18</sup> For each  $\bar{w}_h$ , we first minimise total portfolio variance given a guess on the required portfolio return so that we obtain a few points in mean-variance (or mean-standard deviation) space. These points then trace out the constrained portfolio frontiers. Given  $\gamma = 6$  we find the best point for each  $\bar{w}_h$  by trying all points on the frontiers in the utility function. Using Excel's Solver function we then find the optimal weights on stocks and risk-free *given* all  $\bar{w}_h$ . A detailed description of this procedure is available in Appendix B.

The numerical results are presented in the lower half of Table 2.3. We see that households overall borrow nearly as much as they can, but, since they are credit constrained, have a far more reasonable life-cycle profile than under the unconstrained solution. From looking at Figure 2.5 it is clear that the match with data again is poor: true, households' actual borrowing has much of the same profile as in the model, but there is a significant discrepancy in magnitude. Further, the model prescribes large equity holdings for all  $\bar{w}_h$ , dwarfing the observed allocations. Are people perhaps more risk averse than we think? Are housing risk really crowding out stock holdings (as we mentioned in Chapter 1)? Or, could we find another explanation?

Since our results in the previous section was quite encouraging, we will run through our numerical procedure with the "older" 1966-1991 return estimates as well. The method is still described in Appendix B so we will just report the results here.

From Figure 2.6 and Table 2.4 we see that the match now is close to perfect: optimal stock holdings are still a bit higher than in the data, but the discrepancy

<sup>18</sup>We assume  $\phi = 0$  for simplicity and plausability.

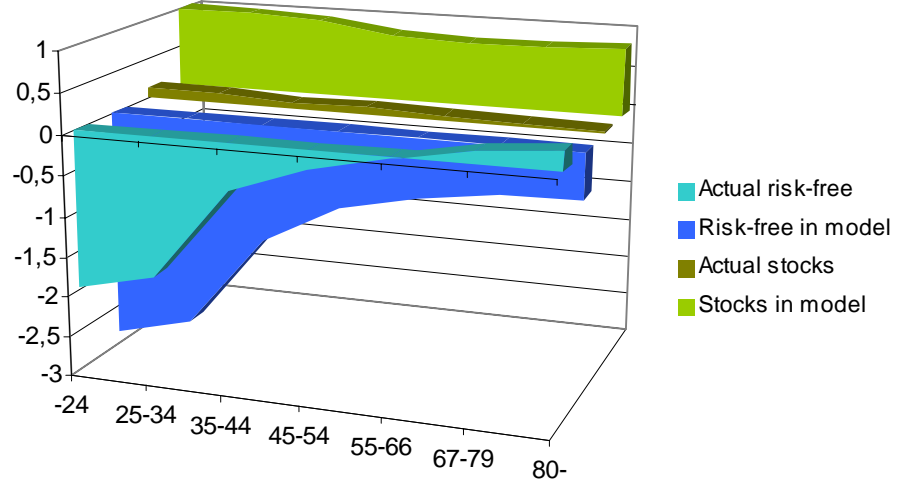


Figure 2.5: Performance of the constrained model with newer estimates by age cohorts. Portfolio shares of the model vs. Norwegian household observations.

is negligible compared to our earlier results, while optimal debt is dead on almost all the way. The fit seems best for the young households. The observed older household portfolios are in a way “safer” since the holdings of stocks are lower and the positive net holdings of risk-free are higher there than in the model. This makes good sense when we remember that we have assumed a relative risk aversion coefficient of 6 for all cohorts, and that our earlier results indicated that older portfolios generally are products of higher degrees of risk aversion.

The life-cycle predictions of the portfolio choice model is thus that

(...) young households with high values of  $[w_h]$  are forced by their high degree of leverage to hold a risky portfolio and therefore

	-24	25-34	35-44	45-54	55-66	67-79	80-
Housing	2.848	2.607	1.492	1.132	0.959	0.801	0.736
Stocks	0.121	0.117	0.077	0.064	0.064	0.037	0.022
Risk-free	-1.969	-1.724	-0.568	-0.229	-0.023	0.162	0.243
$w_s^c$	0.187	0.154	0.157	0.231	0.262	0.154	0.213
$w_f^c$	-2.035	-1.761	-0.650	-0.363	-0.221	0.045	0.051

Table 2.4: Actual portfolio shares (upper half) vs. the prescriptions of the constrained model with older estimates (lower half). The 1966-1991 return estimates are listed in Table A.1 in Appendix A.

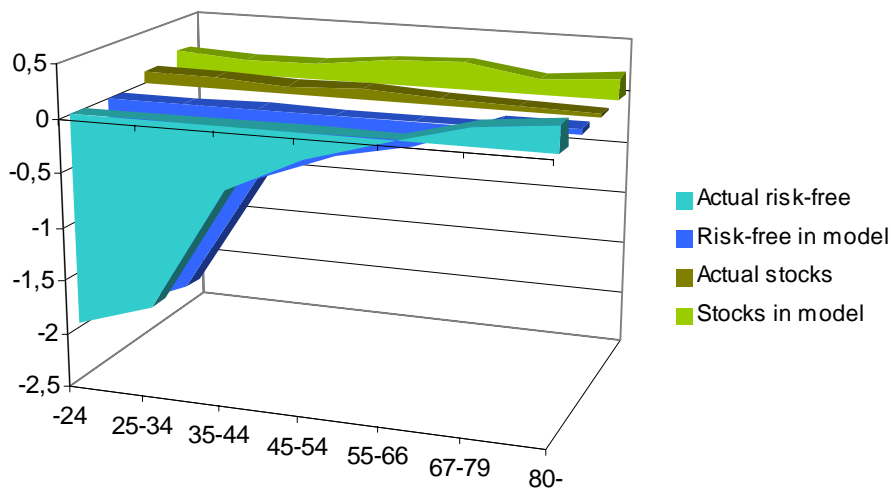


Figure 2.6: Performance of the constrained model with older estimates by age cohorts. Portfolio shares of the model vs. Norwegian household observations.

use their net worth to reduce portfolio risk rather than attempting to further increase their expected return. To reduce portfolio risk, the young households hold only a small to moderate share (...) of stocks, and reduce their leverage, either explicitly by paying down the mortgage, or implicitly by keeping the 100-percent mortgage and simultaneously holding bonds. As wealth accumulates and the value of  $[w_h]$  falls, the household shifts a greater fraction of its financial assets into stocks. (Flavin and Yamashita, 2002:355)

That is, the static model considered in this chapter can, with a few adjustments, explain and justify the compositions of households' portfolios. In Chapter 1 we noticed that these portfolios were much undiversified and contained little equity. Here however, we have seen that, given the pecuniary assumption on housing dividends, such allocations are perfectly rational for households believing in certain asset properties. Trouble is, these beliefs are not consistent with actual asset properties around the time of the portfolio observations. It seems like people either disregard the tremendous returns of the nineties and early noughties as extraordinary and unsustainable, or that they enjoy even greater utility from housing than what we have accounted for in our efforts.

### 2.2.3 The omission of human capital

It should now be mentioned that a central issue in connection to households' risk behaviour and asset holdings has been omitted in the present discussion, namely that of *human capital*. This is innate to the portfolio choice model, but nonetheless unfortunate since we may avoid some major insights. The household members' ability to earn labour income is usually the key determinant in securing consumption goods and other services, and the holdings of real- and financial assets. Naturally, a household with more human capital can borrow more money, buy more housing, and hold more stocks. The income stream is inherently uncertain and it will thus influence the amount of risk one is willing to be exposed to in the housing- and equity market. Cocco (2005) simulates a dynamic life-cycle model and finds that human capital accounts for 87% of young households' total assets and for 21% for the oldest households. As human capital is measured by the expected discounted future value of one's labour income, this is not surprising. An immediate implication is that young households have riskier total "portfolios" from the outset and should thus be *less willing* to invest in other risky assets such as housing and stocks. However,

the ability to get a better paid job or a new job if the current job is lost will probably decrease with age. So risk related to human capital might actually increase with age. If this is the case, willingness to invest financial capital in risky assets should fall with age. (Harding *et al.*, 2004:18)

Yet, the latter paper's results suggest that households do become more risk averse with age; essentially the same conclusion as we arrived at above.

Cocco's paper further divides the simulated households into a high-income group and a low-income group, and finds that the former's human capital share is 53% while the latter's share is 73%. That is, high-income households' expected future income constitutes a smaller fraction of total assets than that of low-income households. This may sound curious at first but has a straightforward interpretation: those enjoying high labour incomes should be more likely to have money "left over" for investments in real- and financial capital and, as wealth accumulates over time, their human capital becomes less dominating in the total portfolio. Poorer households on the other hand, may not be able to set as much aside for the future and are thus forced to maintain a very high human capital share and accept the added uncertainty to their total portfolio. This is consistent with our findings in Table 1.4 above: wealthier households diversify by holding more stocks, while poorer households overall seem to avoid the equity market.

## Chapter 3

# Dynamic life-cycle model

In the previous chapter we used a static model and basically assumed that the household treated every period individually. Regardless of age and income, the household based its investment decisions on expected returns and expected risk only. In the present chapter however, we are interested in how the household behaves over the life-cycle when past decisions matter for current and future utility. How will the allocations in such an environment differ from those prescribed by the static model? Can we perhaps enable ourselves to more accurately explain or rationalise the observed behaviour with an alternative formulation? To these ends we will employ a simple yet powerful dynamic model. The model is standard and follows mostly Adda and Cooper (2003), Rust (2006), Krusell (2004) and Krueger (2005), but we will nevertheless take the time to provide some basic intuition of the central ideas.

### 3.1 Theoretical framework

Suppose that the household lives for  $T$  periods and that it in each period receives utility from regular consumption and from consuming housing services. We denote the period utility by

$$u(c_t, h_{t+1}),$$

where  $c_t$  is consumption at time  $t$  and  $h_{t+1}$  is the amount of housing service available for consumption at time  $t$ . The reason why we have equipped the latter with subscript  $t+1$  is that we in a moment will define this as the amount of housing *bought* at time  $t$ , and *sold* at time  $t+1$ . This of course implies that we assume all households here to be homeowners. Following Cocco (2004) this is justified by further assuming that owning is strictly preferred to renting due to market frictions. We briefly touched upon this issue in the previous chapters and remarked there that these frictions may be identified as inflated rents owed to moral hazard and as tax benefits to the owner-occupier.

Since we are interested in the whole life-cycle we write the expected discounted sum of all period utilities as

$$u_1 = E \left[ \sum_{t=1}^T \beta^{t-1} u(c_t, h_{t+1}) \right], \quad (3.1)$$

where  $\beta$  is the period-to-period discount factor. Equation (3.1) is thus the expected lifetime utility to the household at time 1. If the maximisation of (3.1) is defined as our objective, it is clear that future well-being matters for the optimal adjustment in the initial period and in all subsequent periods. Further, assume that the household faces roughly the same world of options as in Chapter 2, specifically that it can invest in housing and stocks, and hold positive or negative amounts of a risk-free asset. We can then write the time  $t$  budget constraint as

$$c_t + i_t^h + i_t^s + i_t^b = y_t, \quad (3.2)$$

where  $y_t$  is some exogenously given labour income while  $i_t^h, i_t^s$  and  $i_t^b$  are the amounts invested in housing, stocks and risk-free. We will hereafter refer to the latter as “bonds,” but the meaning and interpretation is the same as in the previous chapters. The investments made in the three assets are equal to the differences between holdings in the current- and the next period:

$$i_t^h = h_{t+1} - (1 - \delta)(1 + r_t^h)h_t, \quad (3.3)$$

$$i_t^s = s_{t+1} - (1 + r_t^s)s_t, \quad (3.4)$$

$$i_t^b = b_{t+1} - (1 + r_f)b_t. \quad (3.5)$$

Here,  $\delta$  is the depreciation of the housing holdings,  $r_f$  is still the real risk-free rate, while  $r_t^h$  and  $r_t^s$  are the real rates of return on housing and stocks respectively. Note that  $r_t^h$  now just embodies the capital gain component of the housing return since the consumption benefit is accounted for in the utility function. Inserting Equations (3.3)-(3.5) into (3.2) yields the household's budget constraint:

$$c_t + h_{t+1} + s_{t+1} + b_{t+1} = y_t + (1 - \delta)(1 + r_t^h)h_t + (1 + r_t^s)s_t + (1 + r_f)b_t. \quad (3.6)$$

Together with non-negativity constraints<sup>1</sup> on consumption, housing and stocks, and with the credit constraint mentioned in Section 2.2.2, we formulate the household's lifetime optimisation problem as

$$\max_{c_t, h_{t+1}, s_{t+1}, b_{t+1}} E \left[ \sum_{t=1}^T \beta^{t-1} u(c_t, h_{t+1}) \right] \quad (3.7)$$

subject to Equation (3.6) and to

$$c_t \geq 0, \quad h_{t+1} \geq 0, \quad s_{t+1} \geq 0 \quad (3.8)$$

$$b_{t+1} + (1 - \phi)h_{t+1} \geq 0. \quad (3.9)$$

---

<sup>1</sup>The requirement that regular consumption and housing service consumption should be non-negative is trivial. The non-negativity constraint on stocks however, is to prevent short sales.

That is, the household seeks to optimally allocate its period  $t$  resources between consumption and the three available assets. The “resources” are current period labour income plus the current value of the assets obtained in the previous period. The amount<sup>2</sup> of housing bought at time  $t$ ,  $h_{t+1}$ , is at the household’s disposal for consumption purposes from the beginning of time  $t$  till the beginning of time  $t + 1$ , when it is assumed sold. Naturally, if the household wishes to continue to receive housing services, it must acquire the then desired amount of housing asset at the beginning of time  $t + 1$ .

Note that we, for tractability, abstract from adjustment- transaction- and entrance costs in this setting. This is not particularly realistic but it makes the analysis of life-cycle allocations much easier.

The returns on housing and stocks are still stochastic but now we assume that

$$r_t^h \sim N(\mu^h, \sigma^h) \quad \text{and} \quad r_t^s \sim N(\mu^s, \sigma^s),$$

where the first two moments are known from our earlier discussion. For computational convenience we further assume that the returns are not correlated with each other (even though we found this to be the case in the Chapter 1 estimates).<sup>3</sup>

### 3.1.1 Household preferences

As before we will identify the form of the utility function as that of constant relative risk aversion, but we will also have to find a way to incorporate housing consumption. Fortunately, this is easily done by the Cobb-Douglas aggregator such that period utility can be written as

$$u(c_t, h_{t+1}) = \frac{(c_t^{1-\theta} h_{t+1}^\theta)^{1-\gamma}}{1-\gamma}, \quad (3.10)$$

where  $\theta$  is the weight on housing consumption in total consumption. Notice the difference between this specification and that in Chapter 2: first, we now care directly about consumption rather than indirectly via the portfolio return; and second, the whole distributions of  $c_t$  and  $h_{t+1}$  matter, not just their first two moments.

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<sup>2</sup>The denomination of the variables in the present setup is money value so that, say,  $h_{t+1}$  is the cost of housing bought at time  $t$ . Thus, adjusting the amount of housing does not necessarily imply adjusting the household’s living space: a relative price increase from one period to the next means an increase in the household’s holdings of housing even if their dwelling is the same throughout.

<sup>3</sup>To incorporate covariance between the housing- and stock returns in the numerical procedure below, we would have had to discretise multivariate probability distributions. Although this is not conceptually or technically difficult, it would, without the employment of sophisticated interpolation methods, have required a great deal more computer power (which is definitely a scarce resource for some).

### 3.1.2 Solution concept

The problem in (3.7)-(3.9) generally has a sequence of numbers for all the variables as its solution, but this solution is very difficult, and perhaps even impossible, to obtain analytically through, for example, the Kuhn-Tucker procedure (Rust, 2006). The reason is that the solution to, say,  $h_{t+1}$  depends on the full history of this and the other variables. Keeping track of optimal investment strategies for all possible histories, for all periods of the problem, will be extremely challenging. When we further note that the variables can take on any value and that the states of the world are inherently uncertain, the task immediately becomes impracticable.

Therefore, we will take the approach of *dynamic programming* to get some answers out of our specification in (3.6)-(3.9). A second glance at the latter problem reveals the key to the procedure: the household's problem is essentially the same in every period. That is, at each time  $t < T$  the objective is to maximise remaining lifetime utility given the constraints, and given the resources brought into the current period from the previous period. We define the household's *value function* as

$$v(h_t, s_t, b_t, r_t^h, r_t^s, y_t, t) \equiv \max_{\{h_{t+1}, s_{t+1}, b_{t+1}\}_{t=1}^T} \sum_{t=1}^T \beta^{t-1} u(c_t, h_{t+1}), \quad (3.11)$$

where  $c_t$  is determined by (3.6). Equation (3.11) is then the value of behaving optimally given the left-hand side variables.<sup>4</sup> Since  $h_t, s_t$  and  $b_t$  depend on past decisions they are called *endogenous* state variables. On the other hand,  $r_t^h, r_t^s, y_t$  and time itself do not depend on the household's past actions and are therefore referred to as the *exogenous* state variables.

It can be shown, for instance as in Krusell (2004), that (3.11) implies the *Bellman equation*,

$$v(h_t, s_t, b_t, r_t^h, r_t^s, y_t, t) = \max_{h_{t+1}, s_{t+1}, b_{t+1}} [u(c_t, h_{t+1}) + \beta E v(h_{t+1}, s_{t+1}, b_{t+1}, r_{t+1}^h, r_{t+1}^s, y_{t+1}, t+1)], \quad (3.12)$$

subject to Equations (3.6), (3.8) and (3.9). It is then clear that the household is optimising by investing (and consuming) optimally by deciding on the values of  $h_{t+1}, s_{t+1}$  and  $b_{t+1}$ , the *control* variables, in each period. Consumption follows from the budget constraint it is thus not truly a control variable. The discount factor,  $\beta$ , determines how much weight to put on immediate utility,  $u(c_t, h_{t+1})$ , and how much to put on future utility,  $v(\bullet, t+1)$ . Now, had we known the form of this value function,  $v(\bullet)$ , the solution could have been very straightforward: beginning in period 1, plug in the known state and maximise current and future utility by picking the optimal controls. Sadly, this is not the way of the world. Once again however, we will see that a numerical procedure comes to rescue.

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<sup>4</sup>The indirect utility function (Varian, 1992).



### 3.1.3 Partial equilibrium

Thanks to the *contraction mapping theorem* (Adda and Cooper, 2003), we know that there exists a unique value function that helps solve the Bellman equation (3.12). By first making a guess on  $v(\bullet, t+1)$  we obtain a new value function,  $Tv(\bullet, t)$ , from (3.12). Then we update the right-hand side of the Bellman by setting  $v = Tv$  and find  $Tv$  once again. This updating is continued until the two functional forms on each side of the equality sign converge. That is, we are performing *value function iterations*.

Once we have settled on the value function, the optimal controls are deduced for all possible states of the world. These behaviour recipes are called *decision rules* as they prescribe the optimal actions given any state. The *principle of optimality* ensures that these rules are valid for all periods of the problem (Rust, 2006).

Since we are only considering the economy from the household's point of view, we are looking for a partial equilibrium solution<sup>5</sup> in which the allocations solve the household's problem. Fortunately, the model setup above guarantees that such a solution exist and that it is unique since (i) the utility function is concave and since (ii) the set of possible realisations of the variables are convex (Varian, 1992).<sup>6</sup> It further follows from above that this equilibrium is characterised by a value function and a set of decision rules. Given the state variables, we thus have

$$\begin{aligned} v^*(h_t, s_t, b_t, r_t^h, r_t^s, y_t, t), \\ h_{t+1}^* &= h^*(h_t, s_t, b_t, r_t^h, r_t^s, y_t, t), \\ s_{t+1}^* &= s^*(h_t, s_t, b_t, r_t^h, r_t^s, y_t, t), \\ b_{t+1}^* &= b^*(h_t, s_t, b_t, r_t^h, r_t^s, y_t, t), \end{aligned}$$

which are valid in all periods  $t < T$ . At time  $T$  however, it must be true that  $v(\bullet, T) = u(c_T, h_{T+1})$  as  $v(\bullet, T+1)$  logically have to equal zero.<sup>7</sup> This anchoring down of the  $T+1$  value function is key to finding a solution to the form on the value function via the value function iteration procedure explained above. Further, the decision rules must also ensure that  $s_{T+1} = 0$  since anything else would be either "illegal" or a waste of money. Note that we cannot put forward a similar statement about housing and bonds. The reason is that last period holdings of housing could, and should, be non-negative due to the formulation of the utility function. As a consequence, the household can also be indebted at death (as long as the credit constraint is not violated). Positive holdings of bonds on the other hand, are inefficient and should be barred when we later implement the model.

<sup>5</sup>As opposed to a *general* equilibrium where markets must clear and with prices and allocations solving the problems of both the production- and consumption side of the economy.

<sup>6</sup>Because the CRRA utility function is concave with diminishing utility from consumption, a maximum exists. The convex set assumption simply means that no combination of variable values, sanctioned by the budget- and credit constraint, is impossible.

<sup>7</sup>The household only lives until the end of period  $T$  so any utility delivered in period  $T+1$  is of nil use regardless.

## 3.2 Computation

As the household's consumption-investment problem can only be solved numerically, we will employ the Matlab software package to do the job. The code, interweaved with step-by-step explanations, is available in Appendix B, so the treatment here will be intentionally brief.

### 3.2.1 Time and parameters

For computational convenience we will assume that the household is “born” at age 20 and dies at age 80. This lifespan is then divided into 12 periods, each representing five years of the household's life.<sup>8</sup> The latter assumption of course means that we will have to transform annual parameter values and annual return estimates such that they are representative for longer periods.

We will see below that solving our problem numerically will require as much computer power as possible, so we will elect to keep the labour income exogenous and certain in every period. This measure frees up computer memory and does not hinder an analysis of the average household allocations as such, but it obviously takes us even further away from a realistic representation of the economy. From Statistics Norway's numbers<sup>9</sup> on average household income for cohorts (same division as in Table 1.3) we deduce the average income for *our* cohorts<sup>10</sup> (the 12 periods) and normalise the period income relative to total, average household income. The result is the following stream of time-only dependent labour income:

$$\{y_t\}_{t=1}^{12} = [0.47, 0.75, 0.96, 1.12, 1.23, 1.28, 1.28, 1.22, 1.11, 0.94, 0.71, 0.43].$$

Of course, had we instead assumed a stochastic labour income process (e.g. as in Hu (2005)) the household would presumably compensate for the added risk by holding fewer risky assets and more of the risk-free asset. That is, it would save more (or equivalently, pay down the mortgage faster) to hedge against a possible lower future income stream. We briefly discussed this topic in Section 2.2.3 as well, and we should still read the below results with this caveat in mind.

As we experimented with two sets of asset return estimates in Chapter 2, we find it natural to do the same here. The risk-free rate of return is then 0.02

<sup>8</sup>The notion of a “household” in this setting is admittedly quite abstract and perhaps not immediately tangible...

<sup>9</sup>Table 3 in *Income and Property Statistics for Households 2002*, Statistics Norway. Available online: [http://www.ssb.no/emner/05/01/nos\\_inntektformue/nos\\_d310/nos\\_d310.pdf](http://www.ssb.no/emner/05/01/nos_inntektformue/nos_d310/nos_d310.pdf)

<sup>10</sup>We first assumed that average annual labour income in each Statistics Norway cohort also was the average income at every age within the cohorts so that *our* cohort averages was found simply by association. For example, the average for the 25-34 cohort was assumed to also hold for our two corresponding cohorts, 25-29 and 30-34, and we found the five-year average by simply multiplying by said number of years. Next, we plotted these averages in Excel and fitted a second degree polynomial trend curve. The function of this trend line then gave smoothened average incomes for “ages” one to 12. Finally, we normalised each new cohort average by dividing by overall five-year average household income.

and 0.03 annually for the “older” and the “newer” estimate sets respectively. To obtain five-year period rates we transform these according to  $r_f = (1 + r_{f,a})^5 - 1$ , and thus end up with 0.1 and 0.16 respectively. The discount rate then follows trivially from  $\beta = 1/(1 + r_f)$ .

Statistics Norway reports average housing stock depreciation as around 3% for the 1998-2004 period.<sup>11</sup> Silos (2007) assumes 4.3% while Chambers *et al.* (2005) use 2.2% for owner-occupied housing (both U.S. data). We are therefore comfortable setting the depreciation rate to 0.03 annually so that we obtain  $\delta = (1.03)^5 - 1 = 0.16$  for the five-year periods.

The share of non-housing consumption is set to  $\theta = 0.85$  as Statistics Norway reports average housing service expenditure being 15.7% of total household consumption expenditure in 2004-2006.<sup>12</sup> Housing service expenditure is there defined as actual and imputed rent (not including electricity, heating, maintenance or other costs). This parameter value is also fairly consistent with those in the literature.<sup>13</sup>

The coefficient on relative risk aversion is set to  $\gamma = 5$  as in Hu (2005). This assumption also seems reasonable when we consider our Chapter 2 results. Also with justification in Chapter 2, we set the required downpayment rate to  $\phi = 0$ .

### 3.2.2 State and control space

At the core of the numerical procedure of our choice, value function iteration, lies the *discretisation* of the state- and control variables. In our original problem formulation above it was in a way implied that these variables could take on literally *any* value: there was no limitations imposed on the span of attainable values (apart from the budget- credit- and non-negativity constraints of course) nor on, say, the allowed number of decimals. In order for the computer to calculate the solution to the problem in Equations (3.6)-(3.9), given our numerical strategy, we must approximate these continuous spaces by discrete spaces. This is done by setting upper and lower bounds for each variable's value and by defining a number of grid points between the extremes. All state- and control variables will then be represented by row vectors of length equal to the number of grid points.

As the bounds should not exclude any possible solutions, they will have to be products of some trial and error, and, after playing around with the code of Appendix B for a while, we settle on the following spans:<sup>14</sup>

$$(h_{\min}, h_{\max}) = (0, 8), (s_{\min}, s_{\max}) = (0, 6), (b_{\min}, b_{\max}) = (-6, 6)$$

Note now that these values are only meaningful in relation to the labour income values stated above.

<sup>11</sup>[http://www.ssb.no/emner/09/01/nr/backup\\_2005-12-05/tab\\_1997-2004\\_18.html](http://www.ssb.no/emner/09/01/nr/backup_2005-12-05/tab_1997-2004_18.html)

<sup>12</sup>Source: <http://www.ssb.no/emner/05/02/fbu/tab-2007-09-10-0.html>

<sup>13</sup>Cocco (2004), Flötto (2006), Hu (2005) and Silos (2007) use 0.9, 0.8, 0.7 and 0.8 respectively.

<sup>14</sup>The lower bounds on housing and stocks are intentionally set to zero due to the non-negativity constraints.

Obviously, the approximation will be better the finer the grids are, but the computation can then take an awful lot of time. Had we had only one state- and control variable we could easily have used a couple of hundred grid points, but the sheer size of our problem combined with the limitations of available computer power, forces us to accept a mere seven points.

The credit constraint in (3.9) cannot be imposed by simply restricting the risk-free asset's state space (as it may appear immediately above). Rather, we will employ a *penalty function* that penalises the household financially if it violates the constraint by subtracting a large number from the period utility. That is,

$$\hat{u}(c_t, h_{t+1}) = u(c_t, h_{t+1}) - \text{pen} \cdot \max(0, -b_{t+1} - h_{t+1})$$

where  $\text{pen}$  is some arbitrarily large number. As long as the household respects the credit constraint, the latter term will be zero and thus not affect utility. If the debt is greater in magnitude than the housing investment however, the household will be punished.

### 3.2.3 Approximation of stochastic returns

Following the method described in Adda and Cooper (2003:56), and developed by Tauchen (1986), we can discretise a continuous distribution by dividing its probability density function into  $N$  equally long intervals and then find the means over each interval. The probability of each of these values to be realised is  $1/N$  if we assume the variables to be identically and independently distributed. Generally, we have a normally distributed random variable  $r_t$  with mean and standard deviation  $\mu$  and  $\sigma$ . The points on the distribution where we make the “cuts” are denoted by  $r_i$  for  $i = 1, \dots, N$  and are defined by

$$r_i \equiv \Psi^{-1} \left( \frac{i-1}{N} \right), \quad (3.13)$$

where  $\Psi^{-1}$  is the inverse cumulative density function of the normal distribution. After calculating these in Excel, we can find the mean value of each interval according to

$$z_i \equiv N\sigma (\psi(r_i) - \psi(r_{i+1})) + \mu, \quad (3.14)$$

where  $\psi$  is the probability density function of the normal distribution. Our two stochastic variables,  $r_t^s$  and  $r_t^h$ , will be discretised in this fashion by using  $N = 3$ , the second coarsest approximation possible. The reason for doing this is that the computational burden tends to “blow up” in magnitude when we add more variables to the problem. In an attempt to avoid the dreaded *curse of dimensionality* (Rust, 2006) we therefore seek to do this as simple as possible.

First, the two sets (“older” and “newer”) of annual means and standard deviatons<sup>15</sup> are transformed into five-year values by the standard rules, as is

<sup>15</sup>In Chapter 1 we found the annual mean stock market return to be 0.188. The housing return mean was 0.11 including an imputed rent of 0.04. We will use an annual mean of 0.07

	<i>Annual</i>	<i>Five-year</i>	<i>Annual</i>	<i>Five-year</i>
Newer	$\sigma_a^h = 0.12$ $\sigma_a^s = 0.20$	$\sigma^h = 0.27$ $\sigma^s = 0.45$	$\mu_a^h = 0.110$ $\mu_a^s = 0.188$	$\mu^h = 0.40$ $\mu^s = 1.29$
Older	$\sigma_a^h = 0.11$ $\sigma_a^s = 0.18$	$\sigma^h = 0.25$ $\sigma^s = 0.40$	$\mu_a^h = 0.016$ $\mu_a^s = 0.060$	$\mu^h = 0.08$ $\mu^s = 0.34$
Generally; $\mu = (1 + \mu_a)^5 - 1$ and $\sigma = \sqrt{5} \cdot \sigma_a$ .				

Table 3.1: Transformation of older and newer return estimates from annual figures to five-year figures.

shown in Table 3.1. Next we find the “cuts” from Equation (3.13) and the means from (3.14). The resulting “state space” of the stochastic variables is thus

$$\begin{aligned} \mathbf{z}_{old}^s &= \begin{bmatrix} -0.39 & 0.34 & 1.07 \end{bmatrix} \text{ and } \mathbf{z}_{old}^h = \begin{bmatrix} -0.65 & 0.08 & 0.81 \end{bmatrix}, \\ \mathbf{z}_{new}^s &= \begin{bmatrix} 0.20 & 1.29 & 2.38 \end{bmatrix} \text{ and } \mathbf{z}_{new}^h = \begin{bmatrix} -0.69 & 0.40 & 1.49 \end{bmatrix}, \end{aligned}$$

where the probability for each outcome in all four cases is trivially equal to  $1/3$ .

At this juncture we will issue yet another caveat to be kept in mind when the discussion of results materialise below: the approximation of “newer” stock returns is not fully capturing the fact that returns may be negative. Further, we see that the worst-case outcome for stocks is a rate of return equal to 0.2 while the corresponding risk-free rate of return was set to 0.16. This means that *arbitrage* is possible: the optimising household should borrow as much as it can, invest in stocks, and collect a risk-free profit of at least 4%.

In Chapters 1 and 2 we also talked about the nineties and early noughties being a period of mostly up-and-up in the stock market and that the “newer” estimates might not be very representative for households’ beliefs nor of the “true” nature of equity investment (whatever that is). Therefore, the output presented below should in any case be taken with a pinch or two of sodium chloride.

### 3.2.4 Simulation

After we have iterated over the value function and then found the decision rules for the controls (and implicitly for consumption), we can easily simulate the life-cycle behaviour and holdings of a number of households. As we elaborate on in Appendix B, we elect to simulate 400 “lives” and seek the average allocations of these. Since all households are identical with respect to labour income, initial

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in this chapter to capture the capital component only. The annual mean standard deviation of the stock market return was identified as 0.2. In Chapter 2 we noted that the corresponding 1966-1991 numbers were 0.06, 0.016 and 0.18 respectively.

Since practically all the risk associated with housing return presumably is due to the capital component, we maintain the assumption of Chapter 1 that the standard deviation of housing price appreciation is 60% of that of the stock market.

endowments (zero holdings of all three assets), and decision rules, the differences must stem from the realisations of the stochastic returns. That is, in the first period all households make the same consumption- and investment decisions since their “states” here are identical (no one has nothing!). From period 2 however, we will start to see differences: although everyone invested the same in period 1, only *some* will get the “good” asset return rates in the beginning of period 2. This means that the states now are different so that the decision rules will prescribe different levels of investment for the households. The farther away from the first period we get, the greater will the differences in the simulated households’ holdings be.

This logic may ring true when we think a bit outside the model as well. Suppose for instance that a group of individuals enter household/consumer life with roughly the same endowments and expected human capital at the same point in time. If they all act economically rational<sup>16</sup> and are equally risk averse, they will presumably be close financially at first, but then begin to drift apart. Some might get lucky in the stock market, some might score big on their housing investment, while others might end up submerged in debts.

Naturally, we would contribute this to different overall choices, personalities, preferences, education etc., but also to randomness (or luck) in, among other things, asset returns.

### 3.3 Life-cycle results

The first results from our simulation are graphed in Figures 3.1 and 3.2. The former shows the average life-cycle allocations for the 400 households, together with the certain labour income, given the so-called “newer” asset return estimates. We see a nearly full mortgage for the average household in the first period, and a huge investment in housing. Consumption is practically equal to labour income while stock market investments are zero. In period 2 both housing and consumption are lower (the latter significantly so) and stock market investments are slightly positive. It perhaps seems people are sacrificing a lot of overall consumption early on in order to get into the stock market.

From period 3 however, consumption levels grow tremendously and lie far above labour income from period 5 onwards. Equity investments fall back in period 3 but show steady growth from period 4. The households in the simulation are responsibly paying down their debts over time rather than investing further in other assets, thus exhibiting risk averse preferences. The stocks-risk-free arbitrage mentioned in Section 3.2.3 is thus not fully exploited as the average debt is consistently lower in magnitude than housing worth after period 4.

Note how different this result is conceptually from what we found in Chapter 2. There, the only variables that governed the investor’s decision were the risk and expected return of the market portfolio, i.e., the *state of the market*. Here on the other hand, the household lets its *own* state of the world determine how

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<sup>16</sup>A pretty big “if” though...

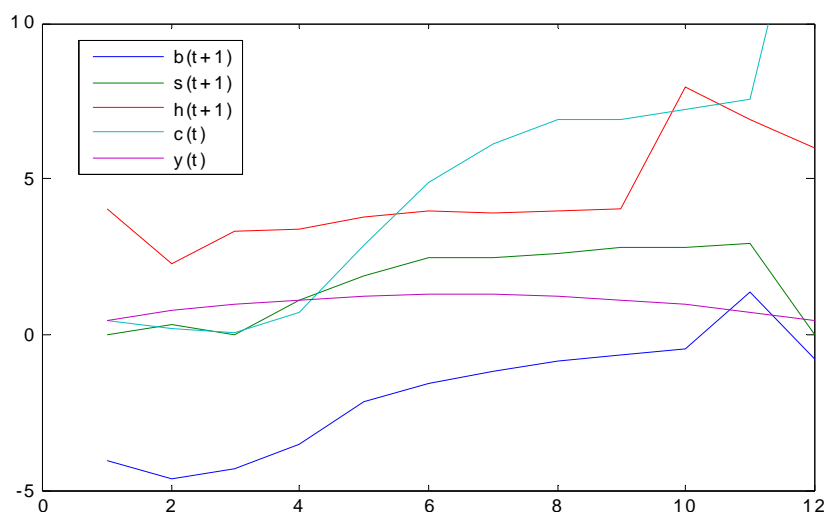


Figure 3.1: Life-cycle allocations with newer estimates.

to invest so that optimal current and future actions are inextricably linked to past decisions and to age. This means that the household can, to a certain extent, control the conditions for optimal resource allocation.

Now, in the final period there is no future so it makes no sense to make provisions for the next period. We see that stock investments here drop to zero and that risk-free actually slips slightly below zero. The latter is OK as long as the housing investment is positive, which it certainly is: since housing consumption is determined by the amount of housing bought at the beginning of each period, the household dies with “unspent” resources as it obviously cannot sell and consume anything post mortem. Note also that non-housing consumption shoots up in the final period as a consequence of the sudden sell-off of stocks. The “unsmooth” stock holding development is perhaps due to high stock market returns; maybe the average household is making too much money on their equity to start the sell-off earlier?

Moreover, we see that, as stock holdings accumulate and debts are paid down, housing stays fairly constant over time (until period 9 that is, when housing shoots up while consumption stabilise). A possible explanation is that households are using the capital gains from their housing asset to increase consumption, pay down the mortgage and invest in the stock market, rather than reinvesting it back into more housing. This makes perfectly good sense in our model, but is it reasonable in the real world? Perhaps not, but it should be possible for a household to take advantage of house price increases by taking out new loans using the house as collateral, thus enjoying relatively low interest

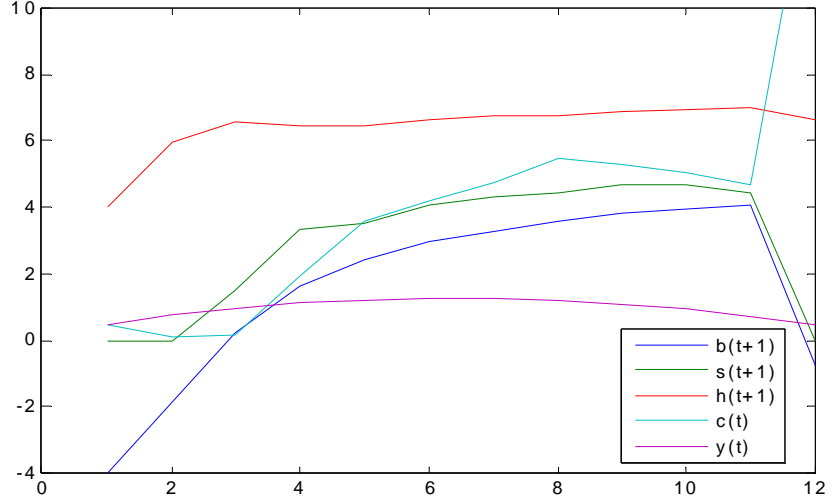


Figure 3.2: Life-cycle allocations with older estimates.

and a chance to make profits in the stock market.

Much of the discussion above is valid for Figure 3.2, which shows the average life-cycle allocations given the “older” return estimates, as well. We see now however, that stock- and housing investments generally are greater than in Figure 3.1. This may sound curious when we remember that the current asset returns are consistently lower than those producing the allocations in the previous figure: why would the households invest more now? In principle, this may answer itself: households save/invest more now just because expected returns *are* lower. In order to attain a certain level of “insurance” against overall return uncertainty, and to push up future consumption, the amounts invested in various assets must be higher when expected returns are lower. On the other hand, lower returns on savings also translate into “today’s” consumption being relatively cheaper than future consumption, thus households should postpone consumption to a greater extent under the newer estimates.

We recognise both these effects in Figure 3.2. Consumption breaks above period income about one period earlier than in the previous figure, but then stabilises at a much lower level. However, the overall picture looks quite odd: how can the households afford so much housing and stocks and still have so little debt? Honestly, the graphed behaviour almost seems to be in violation of the budget constraint itself! If this is true, it would certainly not be a very good attestation of our solution strategy nor of our implementation. Timely then perhaps, to remind ourselves of how coarse our numerical approximation is and that slightly funny-looking results are to be expected. For instance, the distance



between each grid point in the state/control space of the risk-free asset is 2. This is necessary to capture the span of plausible values, but is an extremely coarse approximation when we remember that the level of exogenous labour income only spans from 0.43 to 1.28.

Let us assume the above unpleasantness away for now and briefly consider the holdings of risk-free assets under the older return estimates. Positive amounts of said bonds were not in any way necessary in the situation depicted in Figure 3.1 since the return on stocks never was lower than 20%. With the possibility of negative returns on both housing and stocks however, a risk-averse household would be wise to seek debt abolition at some point. Money in the bank is a potential vital insurance for the households against (un)expected negative shocks to the value of their holdings of stocks and housing. In our framework the latter is particularly important since housing is assumed “rebought” in every period and is an essential component in overall consumption and utility.

Figure 3.2 therefore makes sense in principle, but we see that risk-free turns positive way too early compared to our observations in Chapter 1: while we in Table 1.3 saw the average portfolio share of risk-free turn non-negative around household age 60, the risk-free holdings prematurely shoots above zero at age 35 in said figure.

### Bequest motives

In Section 3.1.3 we wrote that the representative household received no utility after death so that  $v(\bullet, T+1)$  logically had to equal zero. We are not now disputing the absence of post mortem utility but rather suggesting that it may be very reasonable to instead assume that the household draws utility, while alive, from *knowing* that someone else will benefit from the assets they leave behind. That is, the household may very well be interested in *bequeathing* some of its wealth to younger households. Such an assumption has the important implication that  $v(\bullet, T+1)$  then becomes positive: dying with unspent resources (other than housing) yields utility in itself. Let us therefore assume that the household now puts weight on assets left behind according to the CRRA utility function such that

$$v(\bullet, T+1) = \frac{((1-\delta)(1+r_{T+1}^h)h_{T+1} + (1+r_{T+1}^s)s_{T+1} + (1+r_f)b_{T+1})^{1-\gamma}}{1-\gamma}.$$

Figures 3.3 and 3.4 show the modelled life-cycle allocations with such a bequest motive imposed. The latter is virtually identical to Figure 3.2 apart from the absence of equity sell-off at the end of the life-span, and from the lower last-period consumption level. Figure 3.3 is also very similar to its non-bequest twin, but the simulated households here appear to be buying risk-free bonds in the last period. This sounds very reasonable given the data from Chapter 1, but when we remember that the worst-case rate of return on stocks with the newer estimates is 0.2 while the risk-free return is 0.16, the result’s status quickly changes to “curious:” would it not be better to exit life with some (relatively)

low-interest debt thus allowing greater holdings especially of stocks, but also of housing?

Anyway; in the same spirit as our Chapter 2 efforts we will now take a look at how the model's predictions match actual data. In Figures 3.5 and 3.6 we have plotted the life-cycle allocations of Figures 3.3 and 3.4 against observed Norwegian holdings in 2002.<sup>17</sup> In the data, people start out with little housing and debt, and virtually no equity. Housing over the life-span is then nicely hump-shaped with a peak around period 7 (corresponding to age 50-54), while debt increase up to period 4 (age 35-39) when it starts to fall again. From period 9 (age 60-64) risk-free holdings are positive and increasing. Stock holdings are, as we have discussed to great length in earlier paragraphs, negligible throughout compared to the other two asset holdings.

Clearly, the newer estimate allocations lie closer to the actual holdings than those of the older estimates. This is immediately interesting because the portfolio model in Chapter 2 produced the exact opposite conclusion: older estimates were able to *better* explain observed behaviour than newer ones. But, as was the recurring tune in the previous chapter, predicted stock holdings are wildly out of line with the observations.

Although the simulated averages in Figure 3.5 are very jagged and imprecise as such, they are nevertheless matching the smooth, observed holdings in a broader sense. Apart from an initial jump, the housing evolution is not in direct violation with the data; it increases early on, stabilise, and drop down a bit in the latest periods. The model however, recommends postponing the housing peak a few more years and exiting life with a higher housing stock.

The risk-free holdings are perhaps where the model and the data fit best: both evolutions are falling initially, stabilising, and then growing until the end. The differences lie in the magnitudes as the model suggests it is better to borrow more early in life and to bequeath quite a lot more at the end. As always, actual holdings of stocks are nowhere near the predictions of the model.

Now, if we turn the discussion on its head and say that "people are wrong – the model is right," we can conclude that households hold too much housing and too little equity. Of course, this is basically the same remark we ended up with in reference to the Chapter 2 model. Given the present model's results, we thus still observe overinvestment in housing and underinvestment in equity, presumably due to the before mentioned binding housing constraint.

### Credit constraint from labour income

An obvious inconsistency with real world data is that too much debt is allowed relative to households' labour income. While people are typically not granted

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<sup>17</sup>These holdings are those reported in Table 1.3 which again originates from a table similar to Table 1.1. The data is stretched, smoothened and normalised just like the exogenous labour income above. In a nutshell: Statistics Norway's cohort means were "translated" into our 12-period means, plotted in Excel, approximated/smoothened by second degree polynomials (third degree for risk-free for obvious reasons), and finally normalised by average labour income.

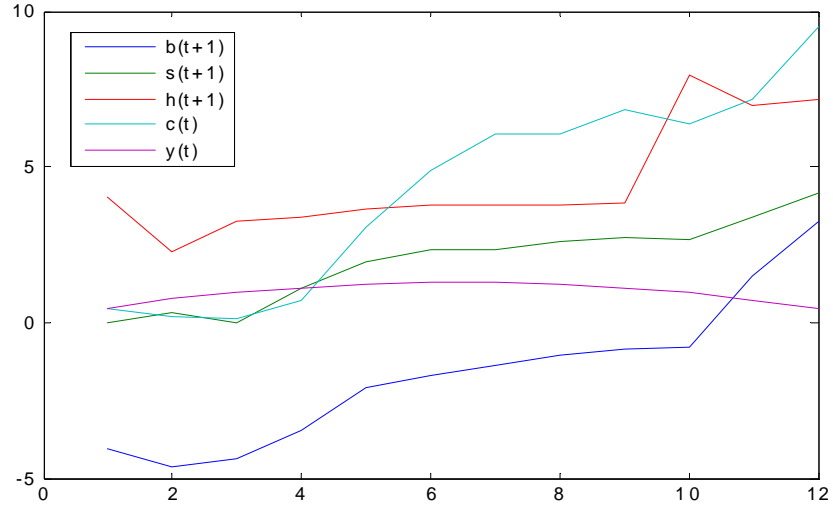


Figure 3.3: Life-cycle allocations with newer estimates and bequest motives.

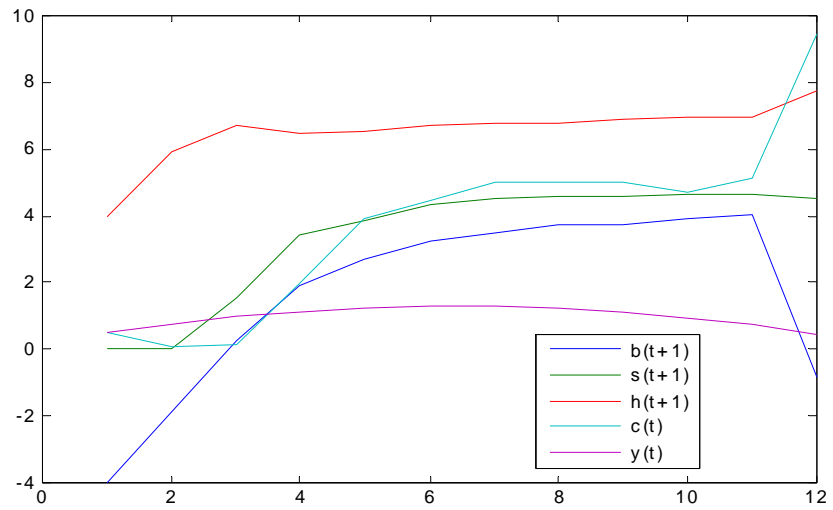


Figure 3.4: Life-cycle allocations with older estimates and bequest motives.

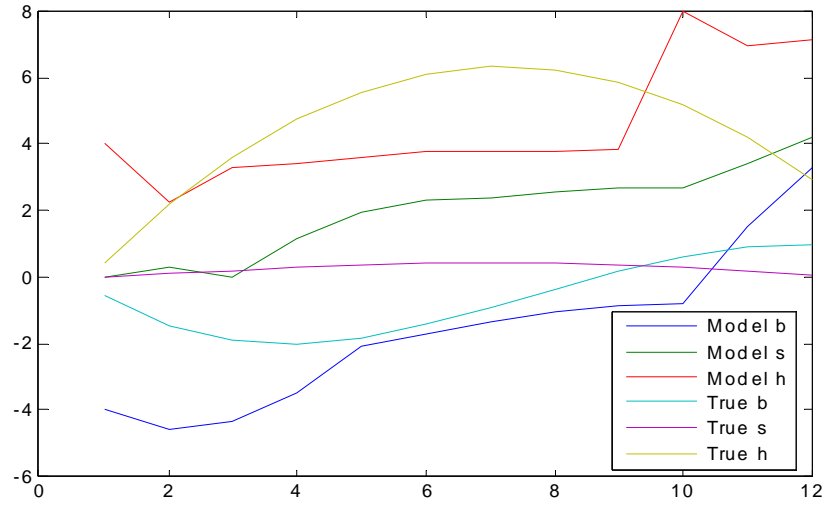


Figure 3.5: Model vs. actual data. Newer estimates and bequest motives.

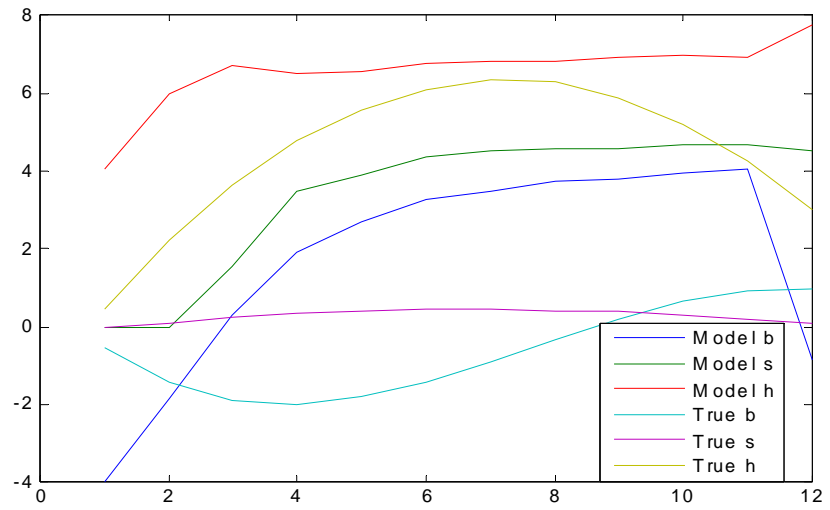


Figure 3.6: Model vs. actual data. Older estimates and bequest motives.

mortgages of magnitude greater than perhaps three or four times their annual income, our model (Figure 3.3) predicts an average debt level of roughly eight times labour income in period 1, and six times labour income in period 2. However, a key simplification in our framework is that labour income is always known and certain. As we have mentioned earlier, this assumption is quite far from reality, but, given such an environment, a restriction on borrowing based on period labour income almost seems “unfair” to the household: after all, the entire human capital value is dead certain and should be taken into consideration when loans are granted.

We will nevertheless perform a quick experiment by replacing the penalty function described in Section 3.2.2 by

$$\hat{u}(c_t, h_{t+1}) = u(c_t, h_{t+1}) - pen \cdot \max(0, -b_{t+1} - 4 \cdot y_t) \quad (3.15)$$

so that household’s are not allowed to borrow more than four times their labour income in any period. Figures 3.7 and 3.8 show the resulting allocations when bequest motives are still assumed to be influencing households’ actions. Oddly, many of our remarks on the differences between Figures 3.1 and 3.2 still apply when we study the differences between Figures 3.3 and 3.7. Most notably, household’s do not seem to “need” mortgages to finance greater housing investments nor equity purchases...

Clearly, something has gone wrong in the computation here as we are getting results that are not making any sense whatsoever. What we would expect when (3.15) is imposed instead of the benchmark credit constraint is of course that housing, equity and consumption should be consistently lower, and that the new credit constraint would be binding. Obviously, the household wants to consume as much goods and housing as possible, and the best way to do this is to borrow as much as possible (at least in Figure 3.7 where the equity return is never lower than the risk-free rate). In our results however, there are only borrowing in the first period and, given that households start out in period 1 with no endowments other than the labour income of that period, the budget constraints are visibly violated.

Had the experiment been successful it would have been most interesting to study the effect of the tougher credit constraint on the consumption-savings trade-off: would stock holdings have had to be given up in order to maintain a high level of housing consumption, or, would the household have had to be content with less housing relative to stocks due to the latter’s profitability? Would perhaps regular consumption be postponed even further so that the household could still enter the stock market?

As we are unfortunately not able to provide a sound explanation of the peculiarities going on in Figures 3.7 and 3.8, we will consider this yet another lesson learnt in the inexplicable ways of the world...

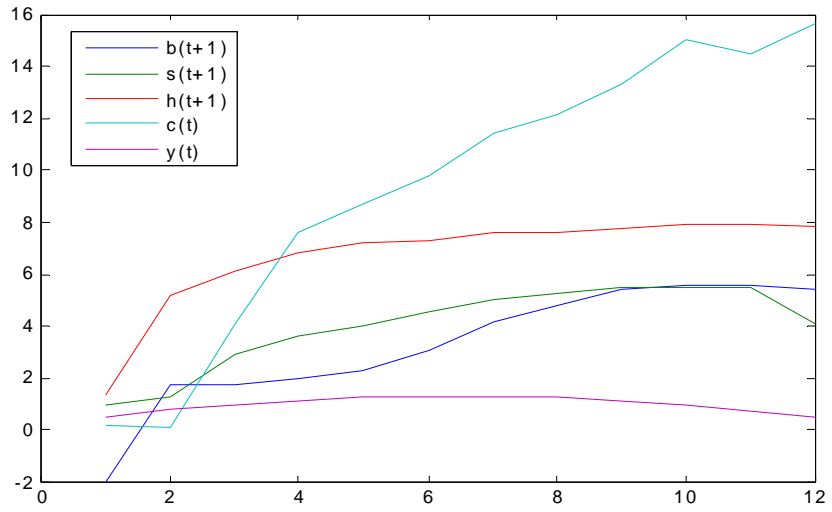


Figure 3.7: Life-cycle allocations with newer estimates, bequest motives and labour income dependent borrowing constraint.

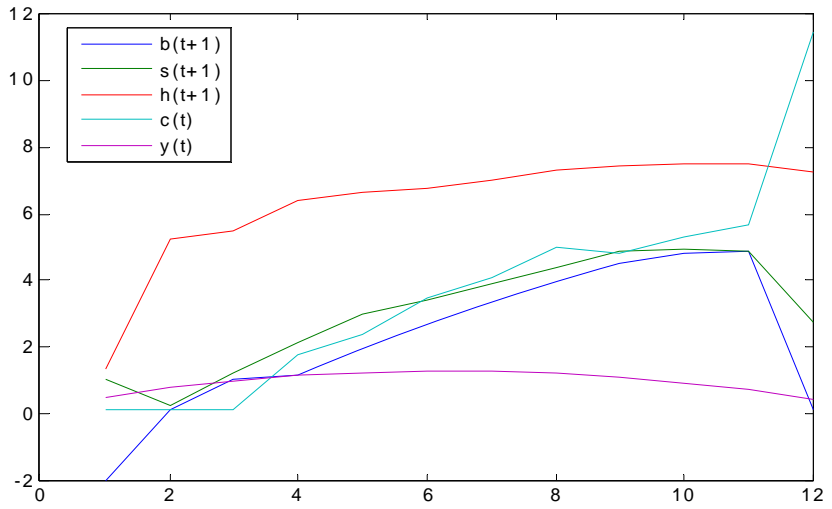


Figure 3.8: Life-cycle allocations with older estimates, bequest motives and labour income dependent borrowing constraint.

## Chapter 4

# Summary and concluding remarks

Housing is typically the greatest investment, and the most valuable asset, of any household. Overall, it dominates the household portfolios and is crucial in the accumulation of wealth over time. Since housing assets can serve as collateral, people are granted large mortgages, and thus even modest returns yield great wealth boosts due to the sheer magnitudes of the investments. Naturally, owning a home also produces housing benefits of great value to the owner-occupier. Everyone needs a place to live and owning a home secures a steady flow of dwelling services that would otherwise have had to be bought in the rental market.

It is no surprise then that Norwegian households' asset portfolios are undiversified: using data from the *Income Distribution Survey 2002* from Statistics Norway, we find that the average portfolio has a housing-to-wealth ratio of 1.175, a stocks-to-wealth ratio of 0.089, and a debt-to-wealth ratio of 0.264. These figures mirror findings in similar surveys of other economies. Breaking the average holdings down to age cohorts shows that young households hold much more housing and debt relative to their wealth than older households. It is also evident that wealth accumulates a great deal over the life-cycle: when population average wealth is normalised to unity, the cohort relative wealth goes from 0.074 for the youngest households, via 2.233 for 55-66 year olds, and finally down to 1.177 for households with head older than 80. The absence of equity holdings are further notable throughout the study: even the portfolios of the wealthiest quartile contain only 10% stocks. Our key motivation is then the apparent question; is this behaviour optimal and can the allocations be rationalised by formal models? If not, what are then the theoretical recommendations?

For our model experiments, the return on risk-free holdings (i.e., bank deposits, loans and bonds) is estimated by an average of Norwegian real bond rates over 1992-2006 less 28% tax, and amounts to a rate of 0.03 annually. Stock return is set to the price appreciation (less 28% tax) plus the dividend rate on the

Oslo stock exchange, less inflation, over the same time period. Housing return is similar as it also consists of a capital gain component and a dividend stream. While the former is easily observed by the house price index from Statistics Norway, the latter is not readily available due to lack of observations. However, with justification in rental data and in theory of the benefit to the owner-occupier, we settle on the before-tax, real risk-free rate as proxy for housing dividend. We further elect to follow advice from the literature on the riskiness of single-home housing investments, and set the standard deviation equal to 60% of that in the stock market. The latter is found to be 0.2 annually for the 1992-2006 period. The reason for not simply using the risk exhibited by the house price index as our measure is that this aggregates market transactions and thus cannot capture the risk of buying a single house. We find mean rates of returns from housing and stocks of 0.11 and 0.188 respectively, with a correlation of 0.33.

Real prices on stocks and housing have appreciated tremendously since the early 1990s. In fact, real prices were fairly stable – or even decreasing – over most of the 20th century, before going through a boom-bust period in the mid-to-late eighties, and then literally taking off around 1993. Since then, real prices have grown by roughly 490% and 260% respectively, and the returns to equity- and house owners have clearly been enormous over these years.

Our first attempt at explaining or rationalising the observed household behaviour mentioned above is the employment of a static mean-variance model of portfolio choice. We assume that the investor only cares about the expected return and variance of the portfolio and that the objective is to minimise this variance given a requirement on the expected return (this is of course equivalent to maximising expected return given some risk tolerance). There are two risky assets, housing and stocks, and one risk-free asset, simply called “risk-free.” The two-fund separation theorem ensures that any investor household will choose a combination of the risk-free asset and the *market portfolio*. While the latter is determined by the properties of the assets available to all investors, each investor will put a weight on the risky- and the risk-free portfolio according to his or her level of risk aversion. The solution to such a setup will then be a set of shares of housing, stocks and risk-free.

With return estimates based on observed asset performances over 1992-2006 we find that the holdings of stocks and housing should be virtually equal in magnitude (the market portfolio composition of risky assets). The less risk averse the household is, the more stocks, housing and debt should it take on. More risk averse households on the other hand, should hold less of the risky assets and positive amounts of the risk-free asset (i.e., bank deposits or bond holdings). Since this result is very far from the observed holdings, we seek an alternative solution by using different return estimates.

Reliable data on equity dividend and on house- and stock prices are available from 1966, so we collect asset properties for the 1966-1991 period and calculate a new set of estimates. The resulting portfolio model solution is uplifting: the framework now prescribes far less stocks relative to housing, which we know is closer to actual allocations. For example, with a risk aversion coefficient equal to 3 we find an “optimal” housing share of 1.129, a share of stocks of 0.484,



and a residual risk-free share of -0.613. This is not very far from the observed, average household portfolio barring the weight on stocks (which is much lower in the data). Further, there seems to be a clear relationship between age and level of risk aversion. That is, for both sets of results, the high-risk aversion portfolios match the observed holdings of older households better, while the model portfolios with lower risk aversion coefficients are better fits with younger households' allocations.

We have mentioned that younger households have the largest shares of housing in their overall portfolio, and that the share tends to decline with age. If we now, as an extension of the applied portfolio model, assume that the amount of housing is *fixed* and determined by the households' demand for housing service consumption, it may be interesting to find the optimal portfolios *given* the housing share of each cohort. That is, what are the optimal shares of stocks and risk-free when the household is already equipped with a certain expected return and risk from the housing holdings?

Unfortunately, our analytical results make no sense in this case due to the absence of a *credit constraint*: the model prescribes outrageously high portfolio shares of stocks and debt. However, by imposing a limitation on how much the households are allowed to borrow ("no more than minus the housing holdings") in a simple numerical procedure in Excel, we obtain far more reasonable results. But, with the 1992-2006 estimates the model recommends an equity share (which is never higher than 0.12 in the data) between 0.75 and 1 for all cohorts and a debt share consistently greater than that in the data. When we instead assume that the 1966-1991 estimates hold, the match with data is almost perfect: low portfolio shares of stocks throughout and a share of risk-free virtually in sync with the Norwegian observations from 2002. It thus seems like people either disregard the tremendous returns of the nineties and early noughties as extraordinary and unsustainable, or that they enjoy even greater utility from housing than what we have accounted for in our efforts. In any event, the older return estimates are better able to explain the observed 2002 holdings than estimates based on asset returns seen in the years around the observation itself.

Our second main exercise is the employment of a dynamic life-cycle model. This approach allows a far richer economic environment to be constructed for the household as we now can consider intertemporal investment-consumption choice and explicitly model the consequences of decisions. It is then assumed that the household receives utility from regular consumption and from housing consumption, and that the objective is, at all stages of life, to maximise the present value of the remaining lifetime utility. While there are still two risky assets and one risk-free asset, we are now adding a stream of labour income which is to be optimally allocated between investment/savings and immediate consumption.

As the lifetime optimisation problem is virtually impossible to solve analytically, we resort to the well-known numerical method of value function iterations. This procedure basically entails approximating an unknown function (the present value of remaining lifetime utility – *the value function*) by iden-

tifying the values of the choice variables that yield the greatest lifetime utility at any point in time, and for all possible past investment decisions. Using this function we next find consumption- and investment rules that prescribe optimal actions given any previous set of allocations and asset returns, and thus solve the household's problem.

It is further assumed that the representative household begins life at age 20, dies at age 80, and that every period in the model corresponds to five years. For computational tractability we abstract from income/human capital uncertainty and simply set the household's income equal to the (normalised) average Norwegian income in 2002 for every period/age.

Since the solution is that of a *representative* household, we simulate the "lives" of 400 individual households, find the average allocations and interpret these as our solution. With the 1992-2006 asset return estimates we find that the household should invest quite heavily in housing from the get-go by taking on the maximum amount of debt (relative to the housing investment), while equity investment and regular consumption are chiefly postponed till age 35. From then on however, consumption levels grow tremendously and lie far above labour income throughout. While equity holdings also grow over the life-cycle and stabilise nicely between housing holdings and labour income, debts are responsibly paid down by the simulated households.

Moreover, as the stock holdings accumulate and debts are abolished, housing stays fairly constant over time. A possible explanation is that households are using the capital gains from their housing asset to increase consumption, pay down the mortgage and invest in the stock market, rather than reinvesting it back into more housing. This makes perfectly good sense in our model, but may not be too reasonable in the real world.

With the 1966-1991 asset return estimates we find mostly the same average profiles except for holdings of stocks and housing being consistently greater than with the 1992-2006 estimates. This sounds a bit curious since the former asset returns are lower than the latter: why would the households invest more now? One rationale is that the amounts invested in various assets must be higher when expected returns are lower in order for the household to attain a comfortable level of precautionary savings, and to be able to push up future consumption. On the other hand, lower returns on savings also means that future consumption is more expensive, thus households should save less and "eat" more.

As our model output shows that regular consumption is pushed forward but attains lower overall levels over the life-cycle compared to the output with the 1992-2006 estimates, both above effects are recognised.

In the final periods of these exercises, equity investments drop to zero since we assume that there is no value in leaving anything behind. But what if we instead say that the household draws utility, while alive, from bequeathing its end-of-life worth to a younger generation? Solving this modified problem we find, as expected, that the simulated households consume less and save more at the end of their lives. Such a specification is of course much closer to reality since it is commonly assumed that bequest motives exist, and because of the inherent uncertainty of the time of death (which we characteristically have abstracted

from in our theoretical escapades).

Finally, we compare the performance of the (bequest) model, with the two sets of estimates, to the observed Norwegian life-cycle holdings, surveyed in 2002. We find that the newer estimates allocations lie much closer to the actual holdings than those of the older estimates. This is immediately interesting because the static portfolio model produced the exact opposite conclusion: older estimates were able to better explain observed behaviour than newer ones. Even though predicted stock holdings are still wildly out of tune with observations, the dynamic model solved with the 1992-2006 estimates can explain the broad features of households' behaviour.

The lesson from both models is essentially *either* that households are holding too little equity and too much housing in their portfolios, *or* that the models themselves are not fully able to account for the high consumption of, and investment in, housing relative to other goods and assets.

Of course, the output of the models depend on the estimated asset return properties, and herein lies the perhaps most interesting result from comparing Chapters 2 and 3: while the framework of the latter produces a better explanation of the 2002 observations with return estimates based on actual returns over 1992-2006, the Chapter 2 model performs better with estimates derived from actual returns over 1966-1991. This would not have been of much value had it not been for the fact that these two sets of estimates are very different. The newer set reflects the tremendous real appreciation over the last 15 or so years, while the older set is more typical for the long-run exhibiting far lower returns and little inter-asset correlation. The dynamic life-cycle model thus appears to offer a more realistic description of household behaviour, given that beliefs and expectations are consistent with observed asset returns. On the other hand, the static portfolio model performs beautifully if the households, for some reason, believe in a less profitable stock market.

However, as the household's holdings of housing are mostly driven by its housing consumption demand, any talk of optimal portfolio composition may almost seem irrelevant simply because there are very few investment options left: given risk averse preferences and a determination on housing consumption, the household has little choice other than paying down debts and maintaining a high share of housing in its total portfolio. If it were to invest in equity rather than abolishing debt, the overall risk would rise regardless of how diversified the list of stocks were. Although the presence of risk from human capital is abstracted from in our models, it can further illuminate the rationale behind the observed behaviour of households: uncertainty regarding future labour income, in addition to the risk from a large housing investment, is deterring equity investment.

As such, the discrepancies between theoretical prescriptions and empirical observations are not easily explained by our models, but the central mechanisms of households' financial behaviour certainly are.



## Appendix A

# Calculations

### The frontier of portfolios

Inserting for the housing asset weight,  $w$ , in the expression for the portfolio variance produces

$$\begin{aligned}
 \sigma_p^2 &= \frac{(\mu_p - \mu_s)^2}{(\mu_h - \mu_s)^2}(\sigma_h^2 + \sigma_s^2 - 2\sigma_{hs}) + \frac{\mu_p - \mu_s}{\mu_h - \mu_s}(\sigma_{hs} - \sigma_s^2)2 + \sigma_s^2 \\
 \sigma_p^2 &= \frac{\mu_p^2 - 2\mu_p\mu_s + \mu_s^2}{(\mu_h - \mu_s)^2}(\sigma_h^2 + \sigma_s^2 - 2\sigma_{hs}) + \frac{\mu_p - \mu_s}{\mu_h - \mu_s}(\sigma_{hs} - \sigma_s^2)2 + \sigma_s^2 \\
 \sigma_p^2 &= \mu_p^2 \frac{\sigma_h^2 + \sigma_s^2 - 2\sigma_{hs}}{(\mu_h - \mu_s)^2} - 2\mu_p\mu_s \frac{\sigma_h^2 + \sigma_s^2 - 2\sigma_{hs}}{(\mu_h - \mu_s)^2} + \mu_s^2 \frac{\sigma_h^2 + \sigma_s^2 - 2\sigma_{hs}}{(\mu_h - \mu_s)^2} \\
 &\quad + \mu_p \frac{(\sigma_{hs} - \sigma_s^2)2}{\mu_h - \mu_s} - \mu_s \frac{(\sigma_{hs} - \sigma_s^2)2}{\mu_h - \mu_s} + \sigma_s^2 \\
 &= \mu_p^2 \frac{\sigma_h^2 + \sigma_s^2 - 2\sigma_{hs}}{(\mu_h - \mu_s)^2} + \mu_p \left( \frac{(\sigma_{hs} - \sigma_s^2)2}{\mu_h - \mu_s} - \frac{2\mu_s(\sigma_h^2 + \sigma_s^2 - 2\sigma_{hs})}{(\mu_h - \mu_s)^2} \right) \\
 &\quad + \mu_s^2 \frac{\sigma_h^2 + \sigma_s^2 - 2\sigma_{hs}}{(\mu_h - \mu_s)^2} - \mu_s \frac{(\sigma_{hs} - \sigma_s^2)2}{\mu_h - \mu_s} + \sigma_s^2 \\
 &= \mu_p^2 \frac{\sigma_h^2 + \sigma_s^2 - 2\sigma_{hs}}{(\mu_h - \mu_s)^2} + \mu_p \frac{(\mu_h - \mu_s)(\sigma_{hs} - \sigma_s^2)2 - 2\mu_s(\sigma_h^2 + \sigma_s^2 - 2\sigma_{hs})}{(\mu_h - \mu_s)^2} \\
 &\quad + \frac{\mu_s^2(\sigma_h^2 + \sigma_s^2 - 2\sigma_{hs}) - \mu_s(\mu_h - \mu_s)(\sigma_{hs} - \sigma_s^2)2 + (\mu_h - \mu_s)^2\sigma_s^2}{(\mu_h - \mu_s)^2} \\
 &= \mu_p^2 \frac{\sigma_h^2 + \sigma_s^2 - 2\sigma_{hs}}{(\mu_h - \mu_s)^2} + \mu_p \frac{-2\mu_s\sigma_h^2 - 2\mu_h\sigma_s^2 + 2\sigma_{hs}(\mu_h + \mu_s)}{(\mu_h - \mu_s)^2} \\
 &\quad + \frac{\mu_s^2\sigma_h^2 + \mu_h^2\sigma_s^2 - 2\mu_h\mu_s\sigma_{hs}}{(\mu_h - \mu_s)^2}.
 \end{aligned}$$

Next, defining

$$\begin{aligned} A &= \frac{\sigma_h^2 + \sigma_s^2 - 2\sigma_{hs}}{(\mu_h - \mu_s)^2}, \\ B &= \frac{-2\mu_s\sigma_h^2 - 2\mu_h\sigma_s^2 + 2\sigma_{hs}(\mu_h + \mu_s)}{(\mu_h - \mu_s)^2}, \\ C &= \frac{\mu_s^2\sigma_h^2 + \mu_h^2\sigma_s^2 - 2\mu_h\mu_s\sigma_{hs}}{(\mu_h - \mu_s)^2}, \end{aligned}$$

enable us to write the parabola function as

$$\sigma_p^2(\mu_p) = A\mu_p^2 + B\mu_p + C.$$

### The tangency portfolio

Inserting Equation (2.3) into (2.2) yields

$$\begin{aligned} \frac{\sqrt{A\mu_T^2 + B\mu_T + C}}{\mu_T - r_f} &= \frac{2A\mu_T + B}{2\sqrt{A\mu_T^2 + B\mu_T + C}} \\ \frac{1}{\mu_T - r_f} &= \frac{2A\mu_T + B}{2A\mu_T^2 + 2B\mu_T + 2C} \\ (\mu_T - r_f)(2A\mu_T + B) &= 2A\mu_T^2 + 2B\mu_T + 2C \\ 2A\mu_T^2 + B\mu_T - 2A\mu_T^2 - 2B\mu_T &= 2C + 2A\mu_T r_f + Br_f \\ -B\mu_T - 2A\mu_T r_f &= 2C + Br_f \\ \mu_T &= -\frac{2C + Br_f}{B + 2Ar_f}. \end{aligned}$$

### Constants and the tangency point

With our estimated parameters we find the three constants:

$$\begin{aligned} A &= \frac{0.016 + 0.039 - 2 \cdot 0.008}{0.006084} = 6.4103 \\ B &= \frac{-2 \cdot 0.188 \cdot 0.016 - 2 \cdot 0.11 \cdot 0.039 + 2 \cdot 0.008 \cdot 0.298}{0.006084} = -1.6154 \\ C &= \frac{0.0353 \cdot 0.016 + 0.0121 \cdot 0.039 - 2 \cdot 0.0207 \cdot 0.008}{0.006084} = 0.1161 \end{aligned}$$

Inserting the constants in Equation (2.4) yields

$$\mu_T = -\frac{2 \cdot 0.1161 - 1.6154 \cdot 0.03}{-1.6154 + 2 \cdot 6.4103 \cdot 0.03} = \frac{0.1837}{1.2308} = 0.1493$$

	<i>Housing</i>	<i>Stocks</i>	<i>Risk-free</i>
<b>Mean</b>	0.056	0.060	0.020
<b>SD</b>	0.110	0.180	0.000

Table A.1: Asset return estimates based on actual returns over 1966-1991. Source: Statistics Norway, Norges Bank, Oslo Stock Exchange and own calculations.

which, from Equation (2.3), produce

$$\begin{aligned}
\sigma_T &= \sqrt{A\mu_T^2 + B\mu_T + C} \\
&= \sqrt{6.4103 \cdot (0.1493)^2 - 1.6154 \cdot 0.1493 + 0.1161} \\
&= \sqrt{0.01781} = 0.1335.
\end{aligned}$$

### The portfolio with historical estimates

Using the numbers in Table A.1, together with the  $\sigma_{hs} = 0$  assumption, we find our three constants to be

$$\begin{aligned}
A &= \frac{0.0121 + 0.0324}{0.000016} = 2781.25 \\
B &= \frac{-2 \cdot 0.06 \cdot 0.0121 - 2 \cdot 0.056 \cdot 0.0324}{0.000016} = -317.55 \\
C &= \frac{0.0036 \cdot 0.0121 + 0.00314 \cdot 0.0324}{0.000016} = 9.073
\end{aligned}$$

so that the tangency point is defined by

$$\begin{aligned}
\mu_T &= -\frac{2 \cdot 9.073 - 317.55 \cdot 0.02}{-317.55 + 2 \cdot 2781.25 \cdot 0.02} = 0.0572 \\
\sigma_T &= \sqrt{2781.25 \cdot (0.0572)^2 - 317.55 \cdot 0.0572 + 9.073} = 0.0946
\end{aligned}$$

The capital market line, as in Equation (2.11):

$$\mu_c = 0.02 + 0.3932 \cdot \sigma_c$$

The weight on housing is

$$w = \frac{\mu_T - \mu_s}{\mu_h - \mu_s} = \frac{0.0572 - 0.06}{0.056 - 0.06} = 0.7$$

so that the overall weight on the risk-free asset is given by

$$w_f = \frac{0.0572 - \mu_c}{0.0372}.$$

Overall weights on housing and stocks are still given by

$$\begin{aligned}
w_h &= (1 - w_f)w, \\
w_s &= (1 - w_f)(1 - w).
\end{aligned}$$





## Appendix B

# Numerical procedures

### Portfolio model with constraints

The following setup is mostly from Chen *et al.* (2008). The expected returns, the variances and covariances are first entered in an Excel spreadsheet as illustrated in Table B.1. Then we load the Excel Solver and impose the following restrictions:

- Cell A6 must be equal to one at all time.
- Cell A4 must always be non-negative.
- Cell A5 must always be greater than or equal to negative A3.

Next, we do seven rounds of optimisation, one for each  $\bar{w}_h$ . In the first round we set A3 to 2.848, in the second 2.607, in the third 1.492, etc. We then tell Solver that cell B7 should be *minimised* for a given value in B9, by varying the values in cells A3, A4 and A5. Of course, since the value in A3 is given, we are only varying over A4 and A5.

For each  $\bar{w}_h$  we try a few different values in B9 and store the corresponding value in B8. This gives us a handful of points on each  $\bar{w}_h$ 's restricted portfolio frontier which we then insert into the mean-variance utility function. By picking the coordinates yielding the greatest utility on each frontier we read-off the portfolio shares from cells A4 and A5.

It should be noted that the approximation producing the results in Chapter 2 was very coarse; only five or six points were identified for each  $\bar{w}_h$ . Further, we had to set all variances and covariances above zero in order to get reasonable results (otherwise we would be advised to invest everything in the risk-free asset). The reason is that we were basically “cheating” a little bit when it came to the risk-free asset because we incorporated it in the hyperbola of risky assets rather than through the capital market line. We therefore sat the risk-free asset's variance, and the covariances that should have been equal to zero, slightly positive.

	A	B	C	D
1		<i>Housing</i>	<i>Stocks</i>	<i>Risk-free</i>
2		$\bar{w}_h$	$w_s$	$w_f$
3	$\bar{w}_h$	$A3 \cdot B2 \cdot \sigma_h^2$	$A3 \cdot C2 \cdot \sigma_{hs}$	$A3 \cdot D2 \cdot \sigma_{hf}$
4	$w_s$	$A4 \cdot B2 \cdot \sigma_{hs}$	$A4 \cdot C2 \cdot \sigma_s^2$	$A4 \cdot D2 \cdot \sigma_{sf}$
5	$w_f$	$A5 \cdot B2 \cdot \sigma_{hf}$	$A5 \cdot C2 \cdot \sigma_{sf}$	$A5 \cdot D2 \cdot \sigma_f^2$
6	SUM(A3:A5)	SUM(B3:B5)	SUM(C3:C5)	SUM(D3:D5)
7	Variance:	SUM(B6:D6)		
8	SD:	$\sqrt{B7}$		
9	Mean:	$A3 \cdot \mu_h + A4 \cdot \mu_s + A5 \cdot \mu_f$		

Table B.1: Spreadsheet setup in Microsoft Excel for solving portfolio optimisation problems numerically.

### Matlab code for the Chapter 3 problem

```

1  % First, any variables or figures from previous sessions are
   removed from the workspace:
2  clear all;
3  close all;
4
5  % Next we define the PARAMETERS of the model. Note that we
   have two sets of estimates, "newer" and "older," but we
   can obviously only use one at a time. Therefore,
   whenever we comment "%N" and "%O" to the right of
   expressions we mean that we have used EITHER %N or %O
   throughout.
6  rf = .16; %N (newer estimate)
7  rf = .1; %O (older estimate)
8  beta = 1/(1+rf); % Discount rate
9  gamma = 5; % Coefficient of relative risk aversion
10 theta = .85; % Weight on non-housing consumption
11 delta = .16; % Housing depreciation rate
12
13 % The "penalty" discussed in the text is an arbitrarily high
   number:
14 pen = 1000000000000000;
15
16 % We use 7 grid points for all controls:
17 gb = 7;
18 gs = 7;
19 gh = 7;
20 % The 7 points are equally spaced between the upper- and
   lower bounds as defined in the first two arguments of
   the "linspace" function:
21 b = linspace(-6,6,gb);
22 s = linspace(0,6,gs);
23 h = linspace(.01,8,gh);

```

```

24
25 % The rates of return on housing and stocks can take on
    three values in the "older" and "newer" sets:
26 rh = [-.65 0.08 0.81]; %O
27 rs = [-.39 0.34 1.07]; %O
28 rh = [-.69 .4 1.49]; %N
29 rs = [.2 1.29 2.38]; %N
30 ph = 1/3; % Probability of each outcome in "rh"
31 ps = 1/3; % Probability of each outcome in "rs"
32 grh = 3; % Number of possible h-outcomes
33 grs = 3; % Number of possible s-outcomes
34
35 % Labour income is certain in all 12 periods:
36 y = [.47 .75 .96 1.12 1.23 1.28 1.28 1.22 1.11 .94 .71 .43];
37
38 % CONSUMPTION AND UTILITY arrays are found by defining c_t
    and u_t for all possible combinations of state- and
    control variables in all time periods. We assign
    counters for all these variables:
39 for a = 1 : 12 % Time counter
40     for ib = 1 : gb % Counter for b_t
41         for is = 1 : gs
42             for ih = 1 : gh
43                 for jb = 1 : gb % Counter for b_t+1
44                     for js = 1 : gs
45                         for jh = 1 : gh
46                             for ns = 1 : grs % Counter for rs_t
47                                 for nh = 1 : grh
48                                     c(a,ib,is,ih,ns,nh,jb,js,jh
                                        ) = y(a) + (1+rf)*b(ib)
                                        - b(jb) + (1+rs(ns))*s
                                        (is) - s(js) + (1-delta
                                        )*(1+rh(nh))*h(ih) - h(
                                        jh);
49                                     if c(a,ib,is,ih,ns,nh,jb,js
                                        ,jh) < 0
50                                         c(a,ib,is,ih,ns,nh,jb,js
                                        ,jh) = .00001;
51                                     end
52                                     u(a,ib,is,ih,ns,nh,jb,js,jh
                                        ) = (((c(a,ib,is,ih,ns,
                                        nh,jb,js,jh)^theta)*(h(
                                        jh)^(1-theta))))^(1-
                                        gamma) - 1)/(1-gamma) -
                                        pen*max(0,-b(jb)-h(jh)
                                        );
53                                 end
54                             end
55                         end
                    end
                end
            end
        end
    end

```



```

86             %W else
87             %W Tv(13,ib,is,ih,ns,nh) = -pen;
88             %W end
89         end
90     end
91 end
92 end
93 end
94
95 % The value function is the sum of instantaneous utility
    and the discounted continuation value, the colons
    represent the controls to be optimally chosen given
    the state variables. Since there are three controls,
    the value function expression must be maximised in
    three operations, one for each choice dimension:
96 for a = 1 : 12
97     for ib = 1 : gb
98         for is = 1 : gs
99             for ih = 1 : gh
100                 for ns = 1 : grs
101                     for nh = 1 : grh
102                         Tv(a,ib,is,ih,ns,nh,,:,:) = max(
                                max(max(squeeze(u(a,ib,is,ih,
                                ns,nh,,:,:) + beta*((1/9)*
                                permute(v(a+1,,:,:,1,1),[4 3
                                2 1]) + (1/9)*permute(v(a
                                +1,,:,:,1,2),[4 3 2 1]) +
                                (1/9)*permute(v(a+1,,:,:,1,3)
                                ,[4 3 2 1]) + (1/9)*permute(v(
                                a+1,,:,:,2,1),[4 3 2 1]) +
                                (1/9)*permute(v(a+1,,:,:,2,2)
                                ,[4 3 2 1]) + (1/9)*permute(v(
                                a+1,,:,:,2,3),[4 3 2 1]) +
                                (1/9)*permute(v(a+1,,:,:,3,1)
                                ,[4 3 2 1]) + (1/9)*permute(v(
                                a+1,,:,:,3,2),[4 3 2 1]) +
                                (1/9)*permute(v(a+1,,:,:,3,3)
                                ,[4 3 2 1]))));
103                     end
104                 end
105             end
106         end
107     end
108 end
109 % The function "squeeze" is to remove singleton
    dimensions so that "permuted" continuation values
    are of the same array dimensions as the utility
    array, thus allowing them to be added together. This
    procedure is the multi-dimension equivalent to
    transposing two-dimensional matrices.

```

```

110
111     % This is the end of "iteration 0" and we now prepare
        for the next round. We first update the iteration
        counter:
112     iter = iter + 1;
113     % Next comes the central mechanism of our solution
        method. First, the difference between the two
        functions are defined as follows:
114     diff = max(max(max(abs(Tv-v))));
115     % where "abs" refers to the absolute value. If "diff" is
        greater than "convcrit" we go back to the top and
        run another iteration AFTER updating the right-hand
        side value function by the result of the previous
        run:
116     v = Tv;
117 end
118
119 disp(iter) % Shows how many iterations were needed for
        convergence
120
121 % DECISION RULES are found by first locating the points in
        the control vectors yielding the highest value element
        in the value function (which is now known thanks to the
        iteration scheme above) for all states of the world, in
        all time periods.
122 for a = 1 : 12
123     for ib = 1 : gb
124         for is = 1 : gs
125             for ih = 1 : gh
126                 for ns = 1 : grs
127                     for nh = 1 : grh
128                         [T1,tmp1] = max(squeeze(u(a,ib,is,ih,
                            ns,nh,.,.,:)) + beta*((1/9)*
                            permute(v(a+1,.,.,: ,1,1),[4 3 2
                            1]) + (1/9)*permute(v(a
                            +1,.,.,: ,1,2),[4 3 2 1]) + (1/9)*
                            permute(v(a+1,.,.,: ,1,3),[4 3 2
                            1]) + (1/9)*permute(v(a
                            +1,.,.,: ,2,1),[4 3 2 1]) + (1/9)*
                            permute(v(a+1,.,.,: ,2,2),[4 3 2
                            1]) + (1/9)*permute(v(a
                            +1,.,.,: ,2,3),[4 3 2 1]) + (1/9)*
                            permute(v(a+1,.,.,: ,3,1),[4 3 2
                            1]) + (1/9)*permute(v(a
                            +1,.,.,: ,3,2),[4 3 2 1]) + (1/9)*
                            permute(v(a+1,.,.,: ,3,3),[4 3 2
                            1])));
129                         [T2,tmp2] = max(T1);
130                         [T3,tmp3] = max(T2);

```

```

131         % The second argument of [...] is the
            LOCATION of the first argument,
            the actual value.
132         bgridrule(a,ib,is,ih,ns,nh) = tmp1(
            tmp3);
133         sgridrule(a,ib,is,ih,ns,nh) = tmp2(
            tmp3);
134         hgridrule(a,ib,is,ih,ns,nh) = tmp3;
135         % The "grid rules" are arrays of the
            optimal control vector elements for
            all possible states of the world,
            including time. The decision
            rules are then simply arrays of
            corresponding values:
136         bdecrule(a,ib,is,ih,ns,nh) = b(
            bgridrule(a,ib,is,ih,ns,nh));
137         sdecrule(a,ib,is,ih,ns,nh) = s(
            sgridrule(a,ib,is,ih,ns,nh));
138         hdecrule(a,ib,is,ih,ns,nh) = h(
            hgridrule(a,ib,is,ih,ns,nh));
139         end
140     end
141 end
142 end
143 end
144 end
145
146 % Consumption follows trivially by inserting the decision
    rules in the well-known budget constraint. Note that we
    still make sure consumption cannot be negative in the
    same manner as above.
147 for a = 1 : 12
148     for ib = 1 : gb
149         for is = 1 : gs
150             for ih = 1 : gh
151                 for ns = 1 : grs
152                     for nh = 1 : grh
153                         cdecrule(a,ib,is,ih,ns,nh) = y(a) + RF
                            *b(ib) - bdecrule(a,ib,is,ih,ns,nh)
                            ) + (1+rs(ns))*s(is) - sdecrule(a,
                            ib,is,ih,ns,nh) + (1-delta)*(1+rh(
                            nh))*h(ih) - hdecrule(a,ib,is,ih,
                            ns,nh);
154                         if cdecrule(a,ib,is,ih,ns,nh) < 0
155                             cdecrule(a,ib,is,ih,ns,nh) =
                                .00001;
156                         end
157                         wealth(a,ib,is,ih,ns,nh) = RF*b(ib) +
                            (1+rs(ns))*s(is) + (1-delta)*(1+rh
                            (nh))*h(ih);

```

```

158         end
159     end
160 end
161 end
162 end
163 end
164
165 % SIMULATION of households over the life-cycle (period 1 to
166 12):
167 for i = 1 : 400 % No. of simulations
168     % First period wealth is zero so we set the counters
169     accordingly:
170     ib = 4; % Fourth element of the b-vector (equals b_1 =
171     0)
172     is = 1; % First element of the s-vector (equals s_1 = 0)
173     ih = 1; % First element of the h-vector (equals h_1 = 0)
174     for a = 1 : 12
175         draws = rand; % "rand" is a random number between
176         zero and one
177         % If rand < 1/3 the stock return rate is equal to
178         that of the first element of rs above. If 1/3 <
179         rand < 2/3 the return is that of the second
180         element, etc.
181         if draws < ps
182             rsstate = 1;
183         elseif ps < draws < 2*ps
184             rsstate = 2;
185         else
186             rsstate = 3;
187         end
188         drawh = rand; % Same as above.
189         if drawh < ph
190             rhstate = 1;
191         elseif ph < drawh < 2*ph
192             rhstate = 2;
193         else
194             rhstate = 3;
195         end
196
197         % Allocations are governed by the decision rules:
198         bt(a+1,i) = bdecrule(a,ib,is,ih,rsstate,rhstate);
199         st(a+1,i) = sdecrule(a,ib,is,ih,rsstate,rhstate);
200         ht(a+1,i) = hdecrule(a,ib,is,ih,rsstate,rhstate);
201         ct(a,i) = cdecrule(a,ib,is,ih,rsstate,rhstate);
202         % Update the individual states by what was invested
203         in the previous period and return to the top.
204         This continues until we have run through all 12
205         periods.
206         ib = bgridrule(a,ib,is,ih,rsstate,rhstate);
207         is = sgridrule(a,ib,is,ih,rsstate,rhstate);

```



```

198         ih = hgridrule(a,ib,is,ih,rsstate,rhstate);
199     end
200
201     % Means of each "age" for all simulations: (note that we
202     % abbreviate by only listing the first and the last "
203     % mean" here)
204     bbar1 = mean(bt(2,:));
205     % through...
206     bbar12 = mean(bt(13,:));
207     % Same for all:
208     sbar1 = mean(st(2,:));
209     sbar12 = mean(st(13,:));
210     hbar1 = mean(ht(2,:));
211     hbar12 = mean(ht(13,:));
212     cbar1 = mean(ct(1,:));
213     cbar12 = mean(ct(12,:));
214
215 end
216
217 % PLOTTING
218 % Vectors of averages:
219 bbar = [bbar1 bbar2 .. bbar12];
220 sbar = [sbar1 sbar2 .. sbar12];
221 hbar = [hbar1 hbar2 .. hbar12];
222 cbar = [cbar1 cbar2 .. cbar12];
223
224 % Observed relative holdings:
225 trueh = [0.44 2.18 3.61 4.74 5.57 6.10 6.33 6.26 5.88 5.21
226         4.23 2.95];
227 trues = [0.00 0.10 0.22 0.32 0.39 0.43 0.44 0.42 0.38 0.31
228         0.21 0.08];
229 trueb = [-0.58 -1.46 -1.91 -2.01 -1.83 -1.45 -0.94 -0.37
230         0.17 0.62 0.91 0.94];
231
232 % We need to plot against time so we define a vector from 1
233 % to 12:
234 tt = linspace(1,12,12);
235
236 % Figure plots:
237 figure
238 plot(tt,bbar,tt,sbar,tt,hbar,tt,cbar,tt,y)
239 q = legend('b(t+1)', 's(t+1)', 'h(t+1)', 'c(t)', 'y(t)');
240 set(q, 'Interpreter', 'none')
241
242 figure
243 plot(tt,bbar,tt,sbar,tt,hbar,tt,trueb,tt,trues,tt,trueh)
244 q = legend('Model b', 'Model s', 'Model h', 'True b', 'True s', '
245         True h');
246 set(q, 'Interpreter', 'none')
247

```

```
241 figure
242 plot(tt,bshare,tt,sshare,tt,hshare)
243 qs = legend('b(t+1)/W','s(t+1)/W','h(t+1)/W');
244 set(qs,'Interpreter','none')
245
246 % Fin
```

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